UNIVERZITA J. E. PURKYNĚ V ÚSTÍ NAD LABEM
Pedagogická fakulta

Usta ad Albim BOHEMICA

rok 2010

ročník X, číslo 1

ISSN 1802-825

Vydává:
katedra bohemistiky pedagogické fakulty Univerzity J. E. Purkyně v Ústí nad Labem

Adresa:
České Mládeže 8, Ústí nad Labem 400 96

Kontakt:
PaedDr. Ivo Harák, Ph.D.
email: harak@post.cz

Editor vydání: Mgr. Lukáš Círus
Technická redakce: Mgr. Lucie Loukotová
Toto číslo recenzovali: Prof. RNDr. Jiří Cihlář, CSc.
Prof. RNDr. Jan Kopka, CSc.

Contents

Introductory word

You are opening the thematic issue of the Usta and Albim Bohemica magazine, which is devoted to the topic of *The Beauty of Mathematics*. The authors of each individual text came from Czech, Poland, Slovakia, and the United States of America or from the Great Britain. As various are the countries the authors came from, the various are topics they are writing about. This issue contains 15 texts that will inform you about various topics, for example: *Verification of Geometric Statements in Dynamic Geometry Environment* by Martin Billich from the Catholic University in Ružomberok. This text deals with the interactions between geometry and Dynamic Geometry Software (DGS) in the teaching and learning of geometry. Other interesting text is *An Investigation Involving Positive Integers* by Jan Kopka from Jan Evangelista Purkyně University in Ústí nad Labem, by George Feissner from State University of New York and by Leonard Frobisher. These authors are to demonstrate investigative approach and creation of conjectures. Other group of authors from Technical University in Liberec (Jaroslav Perný, Jana Hanková, Tereza Nováková and Tereza Votrubcová) deal with the topic of the *Development of Geometrical Imagination of Pupils*. They discuss the possibilities of development of space imagination of primary school pupils. Adam Płocki from the Instytut Matematyki at the Uniwersytet Pedagogiczny Kraków presents his text about the *Expected Time for Tossing Heads and One Demographic Paradox*. The aim of his paper is the determination of the value of expected random variable as a tool of a solid proof. Other chapters deal with many different topics with commons denomination – Mathematics. We can name for instance *How to Change Students' View of the Proof* by Zdenko Takáč, *On Soccer Ball* by Marián Trenkler, *Development of Mathematical Terms in Preliminary Classes* by Jan Melichar or *Application of the Exponential Functions in Testing* by Juraj Butaš and Mária Lalinská, and many others.

I wish you a pleasant discovering *the Beauty of Mathematics*.

Lukáš Círus

Verification of Geometric Statements in Dynamic Geometry Environment

Martin Billich

Catholic University in Ružomberok
Faculty of Education, Department of Mathematics

Abstract. The study of geometry is very important for a mathematical practicians. In this paper we look at the interactions between geometry and Dynamic Geometry Software (DGS) in the teaching and learning of geometry. In addition to basic information about the WinGCLC program the paper offers also an example of using the built-in theorem prover. By building and testing conjectures, students can pursue in advanced mathematical skills.

Keywords. Dynamic geometry environment, geometry constructions, automated geometry theorem proving.

1 Introduction

The goal of the mathematics teacher is for her/his students to engage in advanced mathematical thinking at any grade level. Building conjectures in geometry that lead to formal and informal proof is one example of advanced mathematical thinking ([9]).

The advent of DGS has changed the situation with geometry as a teaching subject. Students may now explore the secret of Euclidean geometry on their own. They may see with their own eyes that this or that theorem is true, and with guidance in fact discover or rediscover many important and elegant results. This will of course not generally constitute proof, but the point is that the insight comes first and then creates the question: why is this so?

In this paper we present basic information of WinGCLC - tool for producing geometrical illustrations for LaTeX, and automated theorem prover (GCLCprover). All constructions and conjectures are stored in formal, declarative representation that can be used as a description of a construction, a description of a figure, and also as a formal description of a conjecture that can be attempted to be proved by the developed theorem prover.

2 Background

In this section we give some basic background information about formal constructions with WinGCLC software, and an application GCLCprover. For more details about WinGCLC system, see [1,3,5].

2.1 WinGCLC - Basic Ideas

WinGCLC package is a tool which enables producing geometrical figures (i.e., digital illustrations) on the basis of their formal descriptions. This approach is guided by the idea of formal geometrical constructions. A geometrical construction is a sequence of specific, primitive construction steps (*elementary constructions*).

WinGCLC uses a specific language for describing figures. GC language consists of the following groups of commands: *definitions*, *basic constructions*, *transformations*, *drawing commands*, *marking and printing commands*, *low level commands*, *Cartesian commands*, *commands for describing animations*, *commands for the geometry theorem prover*. These descriptions are compiled by the processor and can be exported to different output formats. There is an interface which enables simple and interactive use of a range of functionalities, including making animations.

While a construction is an abstract procedure, in order to make its representation in Cartesian plane, we still have to make a some link between these two. For instance, given three vertices of a triangle we can construct a center of its inscribed circle (by using primitive constructions), but in order to represent this construction in Cartesian plane, we have to take three particular Cartesian points as vertices of the triangle. Thus, figure descriptions in WinGCLC are usually made by a list of definitions of several (usually very few) fixed points (defined in terms of Cartesian plane, e.g., by pairs of coordinates) and a list of construction steps based on that points.

2.2 GCLCprover

Automated theorem proving in geometry has two major lines of research: synthetic proof style and algebraic proof style. Algebraic proof style methods are based on reducing geometric properties to algebraic properties expressed in terms of Cartesian coordinates. These methods are usually very efficient, but the proofs they produce do not reflect the geometric nature of the problem and they give only a *yes* or *no* conclusion. Synthetic methods attempt to automate traditional geometry proof methods.

The geometry theorem prover built into WinGCLC is based on the area method (see [7]). This method belongs to the group of synthetic methods. The main idea of the method is to express hypotheses of a theorem using a set of constructive statements, each of them introducing a new point, and to express a conclusion by an equality of expressions in geometric quantities such as *ratio of directed segments*

($\frac{\overrightarrow{AB}}{\overrightarrow{CD}}$), *signed area* ($S_{ABC}$, is the area of a triangle ABC with a sign depending on its orientation in the plane) and *Pythagoras difference* ($P_{ABC} = AB^2 + CB^2 - AC^2$) as a generalization of the Pythagoras equality (for details see [8]). Expressing some common geometric notions using S_{ABC}, ratios and P_{ABC} are given in Table 1.

Geometric notions	**Formalizations**
A, B and C are collinear	$S_{ABC} = 0$
$AB \| CD$	$S_{ABC} = S_{ABD}$
M is the midpoint of AB	$\frac{\overrightarrow{AB}}{\overrightarrow{AM}} = 2 \wedge S_{ABM} = 0$
$AB \perp BC$	$P_{ABC} = 0$
$AB \perp CD$	$P_{ACD} = P_{BCD}$
$A = B$	$P_{ABA} = 0$

Table 1.

The proof is then based on eliminating (in reverse order) the points introduced before, using for that purpose a set of appropriate lemmas. After eliminating all introduced points, the current goal becomes a trivial equality that can be simply tested for validity. In all stages, different expression simplifications are applied to the current goal.

The theorem prover can prove *any* geometry theorem expressed in terms of geometry quantities, and involving only points introduced by using the commands `point`, `line`, `intersec`, `midpoint`, `med`, `perp`, `foot`, `parallel`, `translate`, `towards`, `online` (see [5,7]).

3 Examples

In geometry, we often come across the word *median* while studying triangles. Median of a *triangle* is a line segment joining a vertex of a triangle to the midpoint of the opposite side. There are some basic properties of medians which make them very important in mathematics. They are as follows:

Example 1. Let ABC be a triangle, and let A_1 and B_1 be the midpoints of BC and AC respectively. The crucial point is: do the medians AA_1 and BB_1 intersect at point T which divides medians into a 1 : 2 ratio (with the larger portion toward the vertex and the smaller portion toward the side)?

We can use WinGCLC to answer this question, by describing the construction and proving the property: given three fixed distinct points A, B, C; let A_1 and B_1 be the midpoints of BC and AC respectively. We can construct a point T as the intersection point of the line segments BB_1 and AA_1. The WinGCLC code for this construction and the corresponding illustration (LaTeX output), are shown in Figure 1.

```
point A 1  1
point      1
point C

midpoint A_1   C
midpoint  _1 A C
midpoint C_1 A

intersec T A_1 A  _1

drawse ment A
drawse ment A C
drawse ment   C
drawse ment A_1 A
drawse ment  _1
drawdas se ment C_1 C

cmar _l  A
cmar _r
cmar _t  C
```

```
cmar _r A_1
cmar _l  _1
cmar _t T

prove {equal {sratio A T T A_1 }{2}}
```

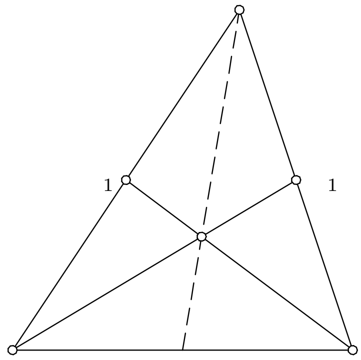


Figure 1.

It can be checked (using GCLCprover) that a point T divides medians AA_1, BB_1 into a 1 : 2 ratio, i.e. $\frac{\overrightarrow{AT}}{\overrightarrow{TA_1}} = 2$. This statement can be given in the code of GCLC language by following line:

```
prove {equal {sratio A T T A_1 }{2}}
```

The prover produces a short report of information on number of steps performed, on CPU time spent and whether or not the conjecture has been proved. For our example we have:

> The theorem prover based on the area method used.
> Number of elimination proof steps: 5
> Number of geometric proof steps: 14
> Number of algebraic proof steps: 25
> Total number of proof steps: 44
> Time spent by the prover: 0.001 seconds
> The conjecture successfully proved.
> The prover output is written in the file medians_ proof.tex.

The prover generate also proof in LaTeX form (in the file `medians_proof.tex`) or/and in XML format. We can control the level of details given in generated proof. The proof consists of *proof steps*. For each step, there is an explanation and its semantic counterpart. This semantic information is calculated for concrete points used in the construction. For our example (on Figure 1.), we will get the following report (a part of the proof):

(1)	$\frac{\overrightarrow{AT}}{\overrightarrow{TA_1}} = 2$	, by the statement
(2)	$\left(-1 \cdot \frac{\overrightarrow{AT}}{\overrightarrow{A_1T}}\right) = 2$	, by geometric simplifications
(3)	$\left(-1 \cdot \frac{S_{AB_1B}}{S_{A_1B_1B}}\right) = 2$	, by Lemma 8 (point T eliminated)
$\vdots$	$\vdots$	$\vdots$
(16)	$\frac{\left(-\frac{1}{2} \cdot S_{BAC}\right)}{\left(\left(-\frac{1}{4} \cdot S_{BAC}\right) + \left(\frac{1}{2} \cdot \left(0 + \left(\frac{1}{2} \cdot (0 + (-1 \cdot 0))\right)\right)\right)\right)} = 2$	, by geometric simplifications
(17)	$\frac{-\frac{1}{2}}{-\frac{1}{4}} = 2$	, by algebraic simplifications
(18)	$-\frac{1}{2} = -\frac{1}{2}$	, by algebraic simplifications

Q.E.D.

Example 2 (Concurrency of Medians of a Triangle). All three medians of a triangle intersect at a point T that is called the *centroid*.

One can ensure that this is valid statement, i.e. point T (intersection point of the line segments AA_1 and BB_1) belongs to the median CC_1 of a triangle ABC. This statement can be given to the prover by adding line

```
prove { collinear C C_1 T }
```

to the code given in Figure 1. The condition of collinearity of the points C, C_1 and T is equivalent using the signed area with $S_{CC_1T} = 0$. In this case the prover produces following short report:

The theorem prover based on the area method used.	
Number of elimination proof steps:	8
Number of geometric proof steps:	22
Number of algebraic proof steps:	55
Total number of proof steps:	85
Time spent by the prover: 0.002 seconds	
The conjecture successfully proved.	
The prover output is written in the file centroid_proof.tex.	

At the end of the main proof all non-degenerative conditions are listed:

– $S_{A_1B_1B} \neq S_{AB_1B}$ i.e., lines A_1A and B_1B are not parallel (construction based assumption)
– $S_{ACB} \neq 0$ i.e., points A, C and B are not collinear (cancellation assumption)

4 Summary

In this paper, we presented advantages of GC language for explicit describing construction in Euclidean plane and built-in theorem prover for automatic verification of some geometric statements. This system for constructive geometry provides an environment for modern ways of studying and teaching geometry at different levels.

References

[1] BILLICH, M. *Program WinGCLC vo vyučovaní kužeľosečiek*. In: Proceedings of Symposium on Computer Geometry SCG2009, Vol. 18, STU Bratislava, 5–11, 2009.

[2] CHOU, SHANG-CHING - GAO, XIAO-SHAN - ZHANG, JING-ZHONG *Automated production of traditional proofs for constructive geometry theorems*. In Moshe Vardi, editor, Proceedings of the Eighth Annual IEEE Symposium on Logic in Computer Science LICS, IEEE Computer Society Press, 48–56, 1993.

[3] JANIČIĆ, P. *GCLC a tool for constructive Euclidean geometry and more than that*. In: Proceedings of International Congress of Mathematical Software, Lecture Notes in Computer Science, vol. 4151, Springer Berlin, 58–73, 2006.

[4] JANIČIĆ, P. *Geometry Construction Language*. Journal of Automated Reasoning, Volume 44, Numbers 1-2, Springer Netherlands, 3–24, 2009.

[5] JANIČIĆ, P.; QUARESMA, P. *System Description: GCLCprover + GeoThms*. In: International Joint Conference on Automated Reasoning. Lecture Notes in Artificial Intelligence, Springer Berlin, 145-150, 2006.

[6] QUARESMA, P.; JANIČIĆ, P. *Integrating dynamic geometry software, deduction systems, and theorem repositories*. In: Mathematical Knowledge Management. Lecture Notes in Artificial Intelligence, Springer Berlin, 280-294, 2006.

[7] QUARESMA, P.; JANIČIĆ, P. *Framework for the constructive geometry*. Technical Report TR2006/001, Center for Informatics and Systems of the University of Coimbra, 2006.

[8] QUARESMA, P.; JANIČIĆ, P. *The Area Method, Rigorous Proofs of Lemmas in Hilbert's Style Axioms Systems*. Technical Report TR2006/001, Center for Informatics and Systems of the University of Coimbra, 2009.

[9] TAKÁČ, Z. *MRP tasks, critical thinking and intrinsic motivation to proving*. Teaching Mathematics and Computer Science, 8/1 (2010), Institute of Mathematics and Informatics, University of Debrecen, 149–168.

Multiplying of Numbers on Fingers

Grzegorz Bryll[1], Grażyna Rygał[2]

[1] Professor Emeritus
University of Opole, Poland

[2] Jan Długosz University in Częstochowa,
Pedagogical Department

Abstract. The article presents mathematical substantiation of the method of multiplying on fingers. We show formulae for the product of numbers not only less than ten, but also from the interval $\langle 10, 15\rangle$ as well as of number larger than 15. The given formulae are supplemented with examples of calculations and figures presenting a scheme of multiplying on fingers.

Keywords. Natural numbers, product of numbers, multiplication, fingers.

Multiplying on fingers

The aim of the article is to justify the method of multiplying on fingers and to familiarize the reader with this method. Such multiplying is mostly used for numbers less than 10, but it is possible to multiply on fingers larger numbers as well. Learning calculating in memory has always been difficult for children. Multiplying on fingers is a very natural way to support such calculating method. We are convinced that it is worth to use it in teaching practice.

For any two natural numbers a and b from the closed interval $\langle 5, 10\rangle$, the evident equality is satisfied:

$$a \cdot b = 10[(a-5)+(b-5)] + [5-(a-5)] \cdot [5-(b-5)]. \tag{1}$$

Using this equality, it is possible to find on fingers the product of numbers a and b as follows: Close $a-5$ fingers of the left hand and $b-5$ fingers of the right hand, then multiply the sum of closed fingers by 10. The number of raised fingers of the left hand is $5-(a-5)$, the number of raised fingers of the right hand is $5-(b-5)$. If you add the product of raised fingers, which equals $[5-(a-5)] \cdot [5-(b-5)]$, to the number $10[(a-5)+(b-5)]$, you receive the product $a \cdot b$ (according to formula (1)).

"·" stands for closed finger, "I" stands for raised finger.

Example 1. Find the product $6 \cdot 8$:

– close one finger of the left hand (. IIII) and three fingers of the right hand (. . . II)

– the sum of closed fingers equals $1+3$, the product of raised fingers equals $4 \cdot 2$

– the product of numbers 6 and 8 equals $10(1+3)+4 \cdot 2 = 48$

Example 2. Find the product $9 \cdot 5$ on fingers:

. I	IIIII
the left hand	the right hand

– the sum of closed fingers equals $4+0=4$

– the product of raised fingers equals $1 \cdot 5 = 5$

– hence $9 \cdot 5 = 10 \cdot 4 + 5 = 45$

There is an alternative way to count on fingers the product of natural numbers a and b from the closed interval $\langle 5, 10 \rangle$, using the following equality:

$$a \cdot b = 5[(a-5)+(b-5)+(a-5)(b-5)+25]. \tag{2}$$

To do this close $a-5$ fingers of the left hand and $b-5$ fingers of the right hand. Then multiply the sum of closed fingers by 5 and multiply the numbers of closed fingers $(a-5)$ and $(b-5)$, i.e. $(a-5) \cdot (b-5)$.
The product $a \cdot b$ equals:

$$5 \cdot [(a-5)+(b-5)] + (a-5) \cdot (b-5) + 25.$$

Example 3. Following formula (2) the product $7 \cdot 8$ can be found as follows:

. . III	. . . II
the left hand	the right hand

– the sum of closed fingers equals $2+3=5$
– the product of closed fingers equals $2 \cdot 3 = 6$
– therefore $7 \cdot 8 = 5(2+3)+2 \cdot 3+25 = 25+6+25 = 56$

The way of multiplying numbers according to formula (2) can be generalised as follows: to count the product of natural numbers a and b from the closed interval $\langle 5k, 5(k+1)\rangle$, $(k = 0, 1, 2, ...)$ it is possible to use the following formula:

$$a \cdot b = 5k[(a - 5k) + (b - 5k) + (a - 5k)(b - 5k) + (5k)^2]. \tag{3}$$

Example 4. To find the product $11 \cdot 13$ we consider the interval $\langle 10, 15\rangle$. Then

$$11 - 10 = 1$$
$$13 - 10 = 3$$

According to formula (3) (for $k = 2$):

$$11 \cdot 13 = 10(1 + 3) + 1 \cdot 3 + 10^2 = 40 + 3 + 100 = 143.$$

The product of natural numbers $10 + p$, $10 + q$ from the closed interval $\langle 10, 15\rangle$ can be found as follows:

- close p fingers of the left hand and q fingers of the right hand,
- multiply the sum $p + q$ by 10,
- add the product $p \cdot q$ and the number 100

Example 5. The product $15 \cdot 12$ can be found as follows:

.....	..III
The left hand	The right hand

- the sum of closed fingers equals $5 + 2 = 7$
- the product of closed fingers equals $5 \cdot 2 = 10$
- $15 \cdot 12 = 10(5 + 2) + 5 \cdot 2 + 100 = 70 + 10 + 100 = 180$

On the basis of formula (3) with $k = 3$ for natural numbers a and b from the closed interval $\langle 15, 20\rangle$, we obtain:

$$a \cdot b = 15[(a - 15) + (b - 15)] + (a - 15)(b - 15) + 15^2. \tag{4}$$

Example 6. According to formula (4), we have for the product $16 \cdot 19$:

$$16 - 15 = 1$$
$$19 - 15 = 4$$

- the sum of closed fingers equals $1 + 4 = 5$

- the product of closed fingers equals $1 \cdot 4 = 4$
- $16 \cdot 19 = 15(1+4) + 1 \cdot 4 + 15^2 = 75 + 4 + 225 = 304$

Instead of using the number $5k$ (equation (3)) it is possible to use the number $10m$. Then formula (3) for numbers a and b from the interval $\langle 10l, 10(m+1)\rangle$ will become ($k = 2l$):

$$a \cdot b = 10m[(a - 10m) + (b - 10m)] + (a - 10m) \cdot (b - 10m) + (10m)^2. \qquad (5)$$

Example 7. Let us count the product $12 \cdot 14$ using formula (5) with $m = 1$:

$$12 - 10 = 2$$
$$14 - 10 = 4$$

$$12 \cdot 14 = 10(2+4) + 2 \cdot 4 + 10^2 = 60 + 8 + 100 = 168$$

Example 8. For the product $17 \cdot 13$ we have:

$$17 - 10 = 7$$
$$13 - 10 = 3$$

$$17 \cdot 13 = 10(7+3)7 \cdot 3 + 10^2 = 100 + 21 + 100 = 221$$

Example 9. Assuming $m = 2$ in formula (5), we obtain for the product $21 \cdot 24$:

$$21 - 20 = 1$$
$$24 - 20 = 4$$

$$21 \cdot 24 = 20(1+4) + 1 \cdot 4 + 20^2 = 100 + 4 + 400 = 504$$

References

[1] KORDIEMSKI B. *Rozrywki matematyczne.* Warszawa: Wiedza Powszechna, 1956.

APPLICATION OF THE EXPONENTIAL FUNCTIONS IN TESTING

Juraj Butaš[1], Mária Lalinská[2]

[1] Catholic University in Ružomberok,
Department of Mathematics, Faculty of Education

[2] Catholic University in Ružomberok,
Department of Foreign Languages, Faculty of Education

Abstract. Importance of students to apply the curriculum. We would like to introduce the application of exponential functions in pedagogy, particularly in testing, and its demonstration within all accessible excel graphs. The application is based on the principles of the Item Response Theory–IRT, evolved by G. Rasch and F. M. Lord. It concerns models which express the probability of a correct response of a person to the item as a function of the latent ability of person's and item properties (difficulty, discrimination and guessing).

Keywords. Exponential function, testing, IRT, odds, probability.

We consider properly motivated students in taking any thematic unit and giving some applications of current life. When we talk about students, we mean university students of teaching specializations and high school students. When we talk about the application we mean the real application, not imaginary one.

We introduce a good application of utilization of exponential functions properties. The application is based on principles of item response theory (IRT), evolved by independent Danish statistician Georg Rasch and psychometrician ETS (Educational Testing Service), Frederick M. Lord. Although this psychometric theory has a wider application, we restrict the application in pedagogy, particularly in measuring of educational results (especially in testing) and its demonstration within the accessible excel graphs (some other appropriate graphic editors may be used). The term of item (also a question or task) means the most elementary part of a test for which the points are allocated.

This theory at its basic level, is based on the idea that the probability of correct response to the item is a function of latent (hidden) property or ability. Formal IRT models use mathematical functions which determine the probability of a discrete outcome, such as the correct response of a person to the item in the case of persons and items. Individual parameter can, for example, represent a *student's ability* or a power of personal approach. Item parameters are:

– difficulty,
– discrimination,
– guessing.

In that case we model the functions $p_{ij} = f(\theta_j, b_i)$, $p_{ij} = f(\theta_j, b_i, a_i)$, or $p_{ij} = f(\theta_j, b_i, a_i, c_i)$, where a_i, b_i, c_i are the item parameters and θ_j is the person's ability. Then in the view of the item, we can talk about one-, two- or three-parameter model.

In connection to the measurement of educational results (in testing), the IRT for tasks connected with the development and improvement of tests is used, to keep the banks of items for testing and comparing difficulties associated with previous and subsequent test versions (for example, to allow the comparison of test results in time), etc. For this purpose, a specific software is used (e.g. Winsteps or RUMM), created by the world's great testing institutions.

These models, as many models in the empirical sciences, are based on the Gaussian normal distribution given by the formula for density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp -\frac{(x-\mu)^2}{2\sigma^2},$$

respectively on the standardized normal distribution, where $\sigma = 1$, $\mu = 0$ and the distribution function $F(x)$.

1 Odds and Probability

We introduce a simple definition of the odds and probability:

Odds is a ratio of the number of phenomenon frequency, which is expected, to the number of phenomenon frequency, which is no*n*-expected.

For example: The odds of tomorrow's rain is 3:1, in short 3 (respectively 2:5, in short 0.4).

Probability is a ratio of the number of phenomenon frequency, which is expected, to the number of all possible phenomenon results.

Thus the probability of tomorrow's rain is 3:4 as well as 0.75 = 75% (respectively $2/7 \div 0.287 \approx 28.7\%$).

Transfers Odds versus Probability

$$\textit{probability} = \frac{\textit{odds}}{1 + \textit{odds}}, \quad \text{respectively} \quad \textit{odds} = \frac{\textit{probability}}{1 - \textit{probability}}$$

Example:
If the odds is 3:1, the *probability* $= \frac{odds}{1+odds} = \frac{3:1}{1+3:1} = \frac{3}{4} = 0.75$
If the probability is 0.8, the *odds* $= \frac{probability}{1-probability} = \frac{0.8}{1-0.8} = \frac{0.8}{0.2} = 4:1$

2 Interpretation of Test Scores

What does the gross score mean? What does the score of 34 mean? The gross score is simply the number of points respectively items. We want to make the conclusions arising from the gross score.
Any point or percentage scores can be transformed into the z-score by the equation:

$$z = \frac{score - average}{st.\, deviation}.$$

The average in the z-score is 0 and the standard deviation is 1, the scale is $\langle -4, 4\rangle$. This is the score used by the Rasch model. It does not sound well if we tell a student that he/she had a negative result in the test, e.g. -0.68 (if his result is less than an average). The international measurements (PISA, TIMSS) use the scale $\langle 0, 1000\rangle$ with a diameter of 500 and standard deviation of 100, which is obtained from the z-score by linear transformation of $\text{PISA} = 100.z + 500$.

3 Unit of the Logit

If the odds $k = \exp z$ and $k = \frac{p}{1-p}$, then the probability $p = \frac{\exp z}{1+\exp z}$.

According to Cramer (2003), the inverse function of this function is a function of logit(p). Then $\text{logit}(p) = \log \frac{p}{1-p} = z$, $p(z) = \log \frac{\exp z}{1+\exp z}$.

The graph of this function is a sigmoid, respectively a logistic curve (see Figure 1).

If the z-score has a normal distribution, respectively certain statistical properties, then it is expressed in logits. *The unit of logit is the central term of the item response theory.*

4 The Theory of Latent Scale

According to the theory of latent scale:

- students and items are placed on the same scale, on the same axis,
- students placed higher on the scale have higher abilities,
- items placed higher on the scale have higher difficulty.

lower abilities	$\leftarrow$	**students**	$\rightarrow$	higher abilities
lower difficulty	$\leftarrow$	**items**	$\rightarrow$	higher difficulty

The probability of success is a function of the difference between the student's ability θ and the difficulty b.

When the ability = 3 and the difficulty = 1 $\Rightarrow$ very high probability.
When the ability = 2 and the difficulty = 2 $\Rightarrow$ the probability of 0.5 or 50%.
When the ability = 1 and the difficulty = 2 $\Rightarrow$ low probability.

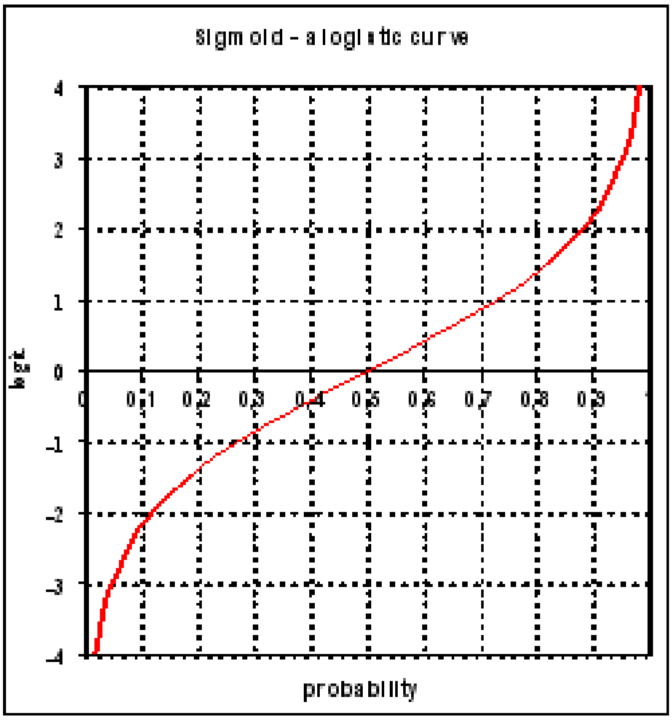


Figure 1. A logistic curve

5 The Rasch Model (IRT, the One-Parameter Logistic (1PL) Model)

The Rasch model describes the parameters of items b_i and the parameters of students θ_n on a common continuum. According to the Rasch model for the n-th student, the probability of the observed score 1 (of the correct response to the item) and the i-th dichotomous item, this relation is available:

$$p(X_{ni} = 1) = \frac{\exp(\theta_n - b_i)}{1 + \exp(\theta_n - b_i)} \tag{1}$$

where θ_n is the student's n ability and b_i is the item i difficulty.

Expressions in the equation (1) have following domains of definition:

Expression	Domain of Definition
$\theta - b$	$(-\infty, \infty)$
$\exp(\theta - b)$	$< 0, \infty)$
$\exp(\theta - b)/(1 - \exp(\theta - b))$	$< 0, 1 >$

A student must have the score 0 or 1 for the dichotomous items. So for each n the probability of the observed score of 0 for dichotomous item i is given by the formula:

$$P(X_{ni} = 0) = 1 - P(X_{ni} = 1) = \frac{1}{1 + \exp(\theta_n - b_i)} \qquad (2)$$

Figure 2 illustrates this model. The horizontal axis shows the latent property that is student's ability θ_n and the vertical axis shows the probability of success in individual items with difficulty b_i.

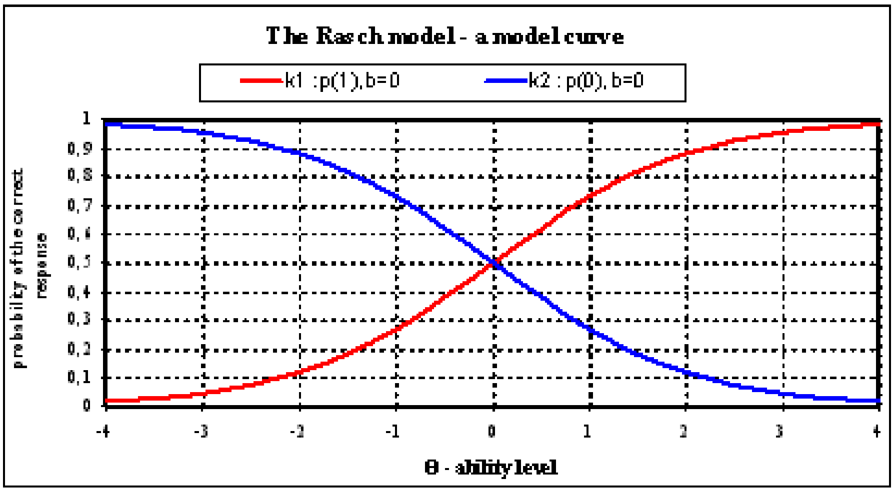


Figure 2. Rasch curves for dichotomous items

The formula (1) is presented by an increasing (red) curve and the formula (2) by a decreasing (blue) curve. It demonstrates a higher probability of the correct response (score 1) close to the higher student's ability (red curve) and a lower probability of incorrect response (score 0) close to the higher student's ability (blue curve). In this dichotomous model, the difficulty b_i corresponds to the ability level in which students reach the probability of correct response exactly equal to 0.5. In this particular case, the difficulty of an item is equal to 0. It is useful to note that this item difficulty corresponds to the point where the both curves intersect each other, where the probability of the score 1 and 0 is the same.

Another figure describes the curves of the probability of three items. IRT names these curves as item characteristic curves (Item Characteristic Curve - ICC), or else, item response functions (Item Response Function - IRF).

In the special case:

b_1, i.e. the difficulty of item 1 is 0 (red),
b_2, i.e. the difficulty of item 2 is -2 (blue),
b_3, i.e. the difficulty of item 3 is 1 (green).

These three curves are expressed by the following equations:

$$k1: \; p = \frac{\exp\theta}{1 + \exp\theta}, \quad k2: \; p = \frac{\exp(\theta - 2)}{1 + \exp(\theta - 2)}, \quad k3: \; p = \frac{\exp(\theta + 1)}{1 + \exp(\theta + 1)}.$$

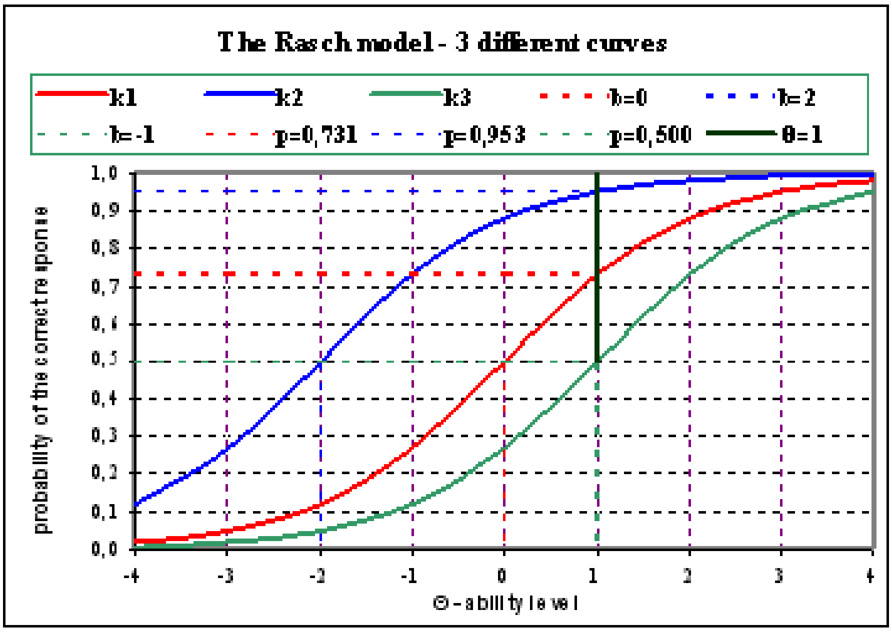


Figure 3. Characteristic curves of three items

We can read from the graphs (of the red, blue and green curve) that the student with an estimated ability $\theta = 1$:

– has the probability of the correct response to the item 1 equal to 0.74,
– has the probability of the correct response to the item 2 equal to 0.95,
– has the probability of the correct response to the item 3 equal to 0.50.

6 The Two-Parameter Logistic (2PL) Model

Within the *two parameters of the logistic*, the 2PL model is equivalent to the 3PL model, where $c_i = 0$. *It will be appropriate for testing of students if the test consists of items with a short open response*, where the probability of guessing is minimal. Model with two parameters a, b is described by the equation:

$$P(X_{ni} = 1) = \frac{\exp a_i(b_i - \theta_n)}{1 + \exp a_i(b_i - \theta_n)}, \tag{3}$$

where θ_n is the n-th student's ability, b_i is the i-th item difficulty, a_i is the i-th item discrimination.

Figure 4 represents a red graph of the model curve, where coefficients of the item are a = 1.2, b = 1; thus the curves with the following equation:

$$p = \frac{\exp[1.2\,(1 - \theta)]}{1 + \exp[1.2\,(1 - \theta)]},$$

where *b* is the item difficulty and it is illustrated by the brown perpendicular to the horizontal axis, and *a* is represented by the green tangent in the point of difficulty *b* = 1.

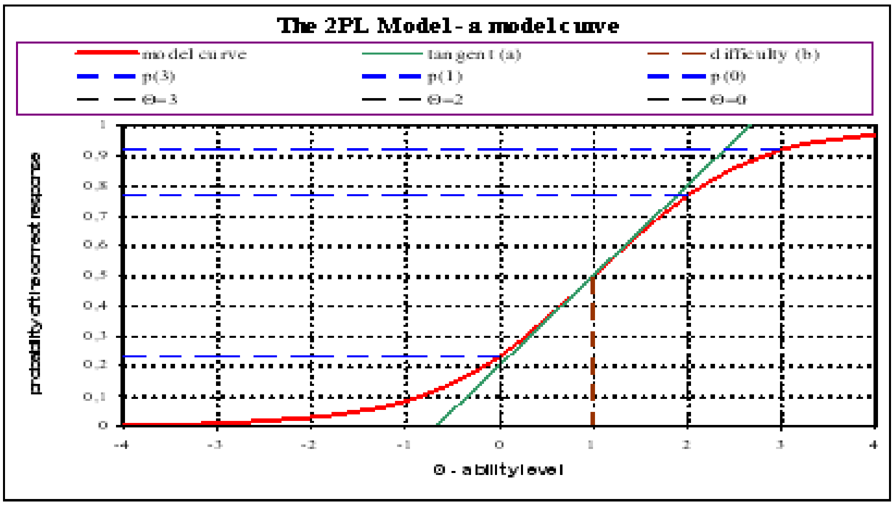


Figure 4. Graph of the 2PL model characteristic curve

Observed levels of ability θ are indicated by the black dashed line, and the blue dashed line indicates a reading (the graph) of the probability of the correct response to an item with these levels of the student's ability θ:

$$\theta = 3 \Rightarrow p = 0.962, \quad \theta = 2 \Rightarrow p = 0.854, \quad \theta = 0 \Rightarrow p = 0.356\,.$$

Figure 5 demonstrates graphs of three characteristic curves of the 2PL model with various values of the discrimination coefficient a = 0.3; 1.2 and 2 with the same difficulty b = 1.

We can see that in a neighborhood of the point of difficulty 1, students with the ability θ = 2 compared to students with the ability θ = 1, are about 0.08, 0.28, 0.39 higher probability of correct response to a question. Respectively the students with ability θ = 0 compared to the students with the ability θ = 1, are about 0.08, 0.28, 0.39 lower probability of correct response to a question.

7 The Three-Parameter Logistic (3PL) Model

Model with three parameters a, b, c is described by the equation:

$$P(X_{ni} = 1) = c_i + (1 - c_i)\frac{\exp a_i(b_i - \theta_n)}{1 + \exp a_i(b_i - \theta_n)}\,, \tag{4}$$

where θ_n is the *n*-th student's ability, b_i is the *i*-th item difficulty, a_i is the *i*-th item discrimination (sensitivity), c_i is a level of the pseudorandom guessing of the *i*-th item.

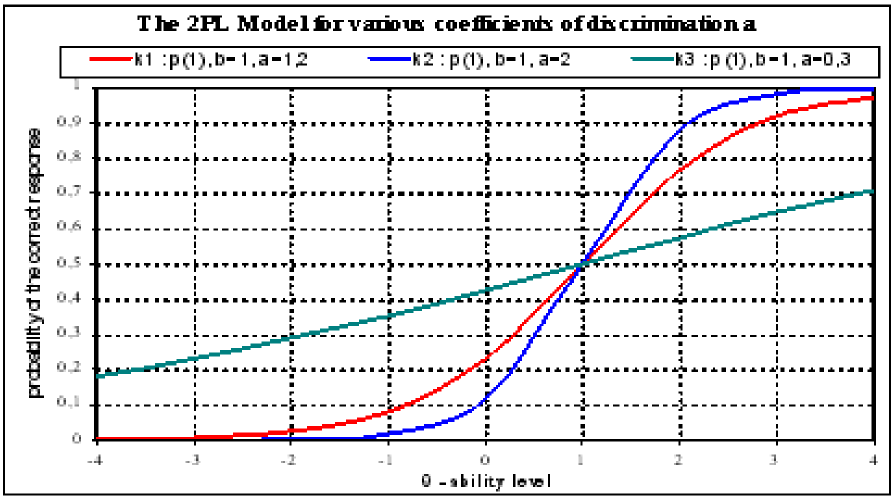


Figure 5. Graphs of three characteristic curves of the 2PL model

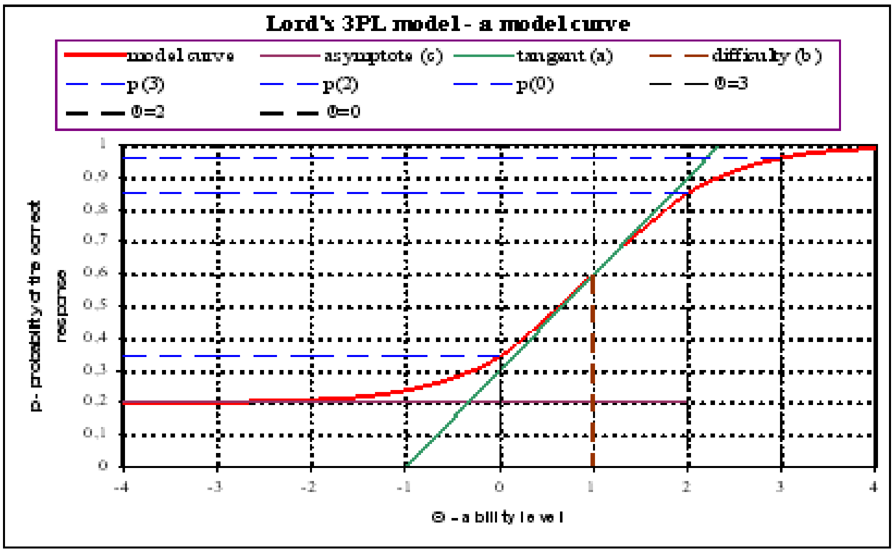


Figure 6. Graph of the 3PL model curve

The figure represents a red graph of the model curve where item coefficients are following $a = 1.5$, $b = 1$, $c = 0.2$, thus the curve done by the equation:

$$p = 0.2 + 0.8\frac{\exp[1.5(1-\theta)]}{1+\exp[1.5(1-\theta)]}$$

b is the item difficulty and it is demonstrated by the brown perpendicular,
a is represented by the green tangent in the point of difficulty $b = 1$,
c is represented by the purple inferior asymptote of the model curve.

Value of the parameter $c = 0.2$ can mean the item with 5 options of response where the probability of correct response is 0.2.

Observed levels of the ability θ are indicated by the black dashed line, the blue dashed line indicates a reading of probability of the correct response to the item (from the graph) within these levels of student's ability θ as well as following:

$$\theta = 3 \Rightarrow p = 0.962, \quad \theta = 2 \Rightarrow p = 0.854, \quad \theta = 0 \Rightarrow p = 0.356.$$

The parameters of an item determine a shape of the ICC and in some cases have a direct geometric interpretation.

Figure 6 shows a geometric interpretation of the parameters on the example of the item characteristic curve of the 3PL model. Here, the parameter b_i represents also the location of the item which, in the case of testing, is denoted as the point of difficulty. It is located at the place where θ of the ICC reaches its maximum slope. This is an example of the medium difficulty item, because $b_i = 0$ is located near the center of the distribution. Note that this model of the scale item difficulty and the concerned student's property have a sign on the same continuum. It is possible to talk about the question as difficult as the level of student's ability. A successful implementation of the items reflects a specific level of abilities. The item parameter $a_i$represents the item discrimination: it means up to what extent the item discriminates between persons in different areas of the latent continuum. This parameter characterizes the slope of the tangent to the ICC when it is at its maximum (inflection point). For example, if the item $a_i = 1,0$, which discriminates students quite well, students with low ability really have less chances to react correctly, than students with a higher ability.

For multi-choice items, the parameter c_i is also used when we consider the estimation of the effects on the probability of the correct response. It means that the probability of the correct response to the item at individuals with very low abilities, is mathematically represented as the inferior asymptote. For example, the item with a choice of four options has 1/4 chance to guess the correct response with an extremely low students' ability and it may have the IRF c_i approximately 0.25. But this assumes that all options are equally plausible, because if one possibility does not have a meaning (it would not be attractive), even the student at the lowest ability level should be able to exclude this possibility.

Figure 7 demonstrates the graphs of characteristic curves of three items in the maximum variability of their coefficients a, b, c – all three coefficients have values which differ by twos.

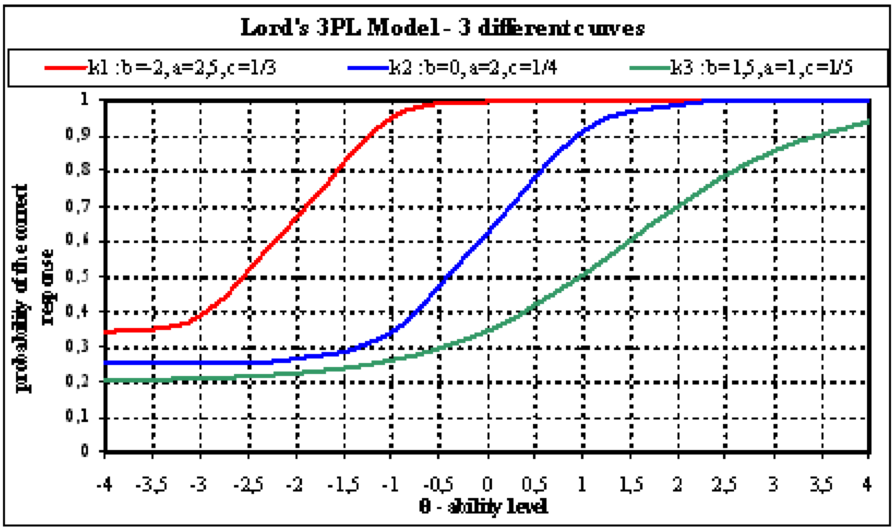


Figure 7. Graphs of three different curves in the 3PL model

In this most general interpretation, the clarity of geometric interpretation has already been steamy, so these cases will not be analyzed within the frame of the exponential functions. Here, we give the priority to more powerful instruments of the IRT – specially developed softwares mentioned in the introduction.

8 Conclusion

We have introduced some mathematical grounds of the item response theory (IRT), which is the application of the exponential functions. We demonstrated some of its geometrically interpretable parts via the applets in an accessible Excel.

Works in the field of the measurement of educational results (of testing) represent, in the last decade, an important majority of works in the worldwide educational research. Teachers themselves in their professional activities often face the testing, also as its implementers. We consider to be necessary and useful to make the students of mathematics teaching courses in the frame of didactics of mathematics, acquainted with that mentioned application.

Since the item response theory is based on many, relatively simple, geometrically interpretable properties of exponential functions, these can serve as examples of applications for high school students.

The above mentioned properties can be demonstrated not only in Excel, but also via other softwares such as many mathematical assistants, computer graphics editors or educational softwares very easily.

References

[1] BAKKER, S. Stupnice, vytváranie stupníc a využitie IRT (teórie odpovedí na testové úlohy). *DutchTest*, Utrecht: ETS Europe (Educational Testing Service), 2005.

[2] BUTAŠ, J. *Meranie výsledkov vzdelávania pomocou testov*. Ružomberok: Katolícka univerzita, 2009.

[3] CRAMER, J. S.*The origins and development of the logit model.*

[4] www.cambrigde.org/resourses/0521815886_default.pdf

[5] LORD, F.M. *Applications of item response theory to practical testing problems*. Mahwah, New Jersey: Lawrence Erlbaum Associated, 1980.

[6] PARTCHEV, I.*A visual guide to item response theory.* Jena: Fridrich-Schiller-Universität, 2004.

[7] RASCH, G. *Probabilistic Models for some Intelligence and Attainments Tests*. Chicago: The University of Chicago Press, 1980.

USE OF ICT IN BRITISH STANDARDS FOR TEACHING MATHEMATICS

Lukáš Círus

Jan Evangelista Purkyně University in Ústí nad Labem,
Faculty of Education, Department of Mathematics and ICT

Abstract. The contribution shows how the British standards on ICT appropriately promote the teaching mathematics to students of younger school age.

Keywords. Teaching mathematics, ICT, Key stage 1, Key stage 2.

1 Introduction

The paper focuses on the British standards in the field of ICT and their influence on Mathematics curriculum, especially the part that involves students of younger school age.

Particular areas, which are an inspiration for teaching, correspond to the Czech first grade of elementary schools. Areas are divided into two groups: Key stage 1 and Key stage 2. These two consist of units or, let us say "lessons" so the teacher is guided and has certain hints when working with students. Each unit has its own name followed on by brief thematic annotation. Then, each specific unit is divided into sections; the introductory section deals with the subject matter and following sections contain the set of short and already aimed exercises. Sections mentioned above are structured; they always start with the main goal of their task, followed by concrete activities and its expected results and the conclusion includes the footnotes for the teacher. Lessons are interconnected and there are links to selected lessons, which can further develop and use the activities. The "goals" and the activities are structured into three groups: for the majority of students, the slow students and for the talented students.

Then, I have chosen individual lessons, which overlap to Mathematics and I notice their brief annotation to approach their content and show a different perspective of teaching ICT in our conditions.

I leave the numbering as it is in the original document; freely available on [1].

2 Key Stage 1

It corresponds to the teaching on the turn of the pre-primary and primary levels of the Czech school system (the last year of kindergarten and 1st grade of elementary school in the Czech school system). It is divided into 10 key phases (1A-2E).

2.1 The selection of the themes of the 1st grade and its summary

Introduction to modelling 1A/ Introduction to simulation 1A

The first key stage is called Introduction to modelling. Pupils encounter the fact the computer can simulate real phenomena they know from surrounding areas. For example, dressing a teddy bear, they try to choose items from the menu and handle them and they are able to understand that ”computer things” are in 2D while those same in reality are in 3D. They learn how to cope with simple didactic adventurous games. Marking and grading information 1D.

In this section, students learn how to organize information into groups due to the criteria given. Again, they use a word processor and word bank. Students learn to describe specific objects by using key words and sort them.

Understanding of instructions and what happened 1F

This section is focused on the instructions and guidance. Students learn the importance of being accurate while choosing the best answer. They learn to work according the instructions and to create their own manual. Students learn about the possibilities of controlling things such as VCR, TV, home appliances and toys via specific controlling devices. They learn to know and configure (program) home appliances to carry on work they are used to. The conclusion of this part can be the importance of making right steps in the right order.

2.2 The selection of the themes of the 2nd grade and its summary

Ways of the “Floor turtle” controlling (simple robot) 2D

Students learn, with the help of a programmable robot, how to place an order and so that the robot will move due to their task. They gain the knowledge that robot is not a person but it has the ability to respond exactly asked instructions. Students learn to place the orders by using numbers as well, eg. 5 step right, 3 steps straight ahead and 1 step left and they check whether the robot has moved due to their instructions. The last idea is that the instructions can be repeated so the students try to programme eg. “staircase” move of the floor or square move.

Tasks and answers 2E

The final chapter of the pre-school teaching is focused on the information and its evident value. That means that the information can be hidden in the graphs, they have limitations and can not answer to all tasks. The chapter then deals with tasks and questions- especially YES/NO questions. It explains the term “database” and the way of gaining information from it.

3 Key stage 2

It corresponds mainly to the primary school teaching in the Czech republic. (2^{nd} – 5^{th} grade of the elementary school). This stage is divided into 18 key steps. (3A-6D).

3.1 The selection of the themes of the 3^{rd} grade and its summary

Introduction to 3D databases

Students learn how to collect and store information. They learn to use the databases as a source of answers to tasks, they also learn to compare paper documents with computer-made databases. This procedure includes first of all students’ ability to work with so called “record cards”, which store the information (eg. Amount of the animal species in the Zoo). They learn the importance of storing information in well-organized way so as the only way how to gain the answer as quickly as possible. Then, they are introducing the electronic version of the “record cards” in PC usage, they learn how to record them, how to get the answers and how to create simple graphs.

3D simulations viewing

In this lesson, students learn how to work with computer simulations, they are to distinguish between the real-world simulations from the imaginary ones, they also learn to predict the simulated events and assess their feasibility. Students will understand the benefits of those simulations, eg. For pilot-training, or ability to change and control of plantsbehavior. It takes a great advantage – they can have a look into the world they would not have chance to in the common school conditions. The students discussion about virtual reality is greatly emphasized.

3.2 The selection of the themes of the 4^{th} grade and its summary

Developing of the work with pictures via the repetition of procedures 4B

Students create a background in the graphical editor, they use a reccuring themes and stamps. They use brushes- they change the size of the print, and they draw using ”dotting” the area. When they want to make a wall paper, they learn to create their own graphic project, the copy it and use specific parts of the plain. They learn by mirroring effect- symmerty- to create different patterns, such as lace or decorated carpets. They gain the knowledge how to save and again check their already saved work.

Divarication of the databases 4C

Students learn to understand more complex databases, not just YES/NO divarication, and they also learn to distinguish those systems and applicate the classification of plants,animals or musical instruments. They will try to make their own YES/NO system, there would be the conclusion of sorting animals into groups; first of all, they will put it on the paper.

Collection and presentation of information – questionnaires and 4D graphs

In this part students learn how to collect information in that form so they can easily work with them, they also learn to create graphs which help to present those information credibly.

First of all, students become familiar with many types of graphs and discuss their usage and informational value. Then, they will try to create a school database of students, they would be lead to discussion about the most effective data collecting way of gaining information of students. The conclusion of this discussion should be their own idea about making a questionnaire with simple questions,so the other students can respond them; through this questionnaire the structure of database would be created.Such questions as “Is there any balance in the answers given by boys and girls? Are there any contrasts among data, which had been collected?” Teacher then leads students to the idea, that when they want to compare data, they should use graph. And also, he teaches students to appoint the hypotheses, and to use the graphic verifications.

At the end of this part, students try to put down into a table eg. the lenght of sunlight during the day, or the height of plant- and depending on the time, they are to find adequate graph for these data. They learn the “doughnut” charts are not able to note all changes in the trnasparent way; so students can see the line charts are more useful. (X,Y).

Phenomena modeling on the screen 4E

Students can compare what they have already known about programming a robot, which moved over the floor, with programming a virtual robot, which moves on the screen. They should realize that the expanse and steps taken on the screen are smaller than on the real ground. They will gain the information the same language is used when programming the virtual robot. It is necessary for students to acquitant them with the orientation within space on the screen; they will soon meet the fact the robot on the screen is able to respond their instructions immediately, unlike Floor Turtle, which responded only after the instructions had been sent. Students will be aware of the reality, that unless they order their instructions as precise as possible, they get back error response. They will know that simple steps can me multuplied by their repetition.

3.3 The selection of the themes of the 5th grade and its summary

Graphic modeling 5A

Students learn how to create and manipulate with the objects in the graphic editor.

They solve tasks, such as drawing the classroom map, while 2D shapes represent the real objects. When manipulating them on the screen, they try to design new lay out of the furtniture in their classroom. They should discuss its pros and cons.

Data analysis and asking questions by using complex look-up 5B

This part concerns students' ability of gaining information from complex databases; their task is to transfer them as graphic data, also printing them and use them so they can gain necessary answers.

Students start by looking-up information eg. about the universe, sort planetsinto groups either if they have planets or due to their size.

In databases, which are prearranged, students rank data according to the marks, "major","minor" or "equals". Then, they learn to find specified data based on more than one criterion (eg. planets with the atmosphere and at least one planet), and they work with logic operators "AND" and "OR".

Information scoring, accuracy and veracity checking 5C

Thich chapter focused on students' ability to score gained information and also find mistakes. They discuss on usage and function of databases in everyday life (eg. school agenda, hospital, evidence of population number); they solve the use of databases and possible faults in saved data.

It is the advantage to make students ready for the instructional databases, which includes faults and inaccuracies, so the main task in this case for students would be to detect them and discuss possible negative effects. Its is useful to teach students work with data while using graphs – they can find anomalies students should think about, find them and clear them up.

Introduction into spreadsheet program 5D

Students use the spreadsheet program and other simple figures for calculations of expenses,eg.of school trip. They understand the advantages of the possibility to only change the inputs and then, expenses would always be re-calculated. First of all,they learn to put data into cells, then how to put figures into cells (adding, subtraction, multiplying and subdividing). Slowly they can move from simple figures to more complex, using brackets. The maint point of this chapter is using of the "SUM" function.

3.4 The selection of the themes of 6th grade and its summary

The spreadsheet program modeling 6B

In this part, students will use the spreadsheet program and think over its changes depending upon variable quantities.

First of all, it would be handy to recall students their previous experience with tablets, then teacher and students start to investigate the mathematical issues with the help of the spreadsheet program. Students will try to formulate figures on their own,that will help them tasks given. For example, they solve the task concerning the girth and content of rectangle, so they can follow how changes of sides has impact on extent and circuit. Students learn to copy mathematic figures, what makes their work even easier to count another values.

The main point of this chapter is work with any of more complex mathematical functions; students create a tablet and by using a figure they gain results and then, they make a graph (for example: $y = x^2$, $y = 2x$, $y = x + 3$).

Operating and monitoring – "What happens, if. . . " 6C

This chapter is concerned about students and their learning of using input of computer for operating. They insert light, compressive and thermal sensors, which helped them to coordinate programmed outputs (eg. when it is getting dark, the light switches on).

Students think and discuss about the difference between the event,that was caused by the change of physical conditions and the event that has been timed. They brainstorm about the savings- if the timed or thermal heating-up of flat of is better and they also think about the combination of both possiblities. Students learn to give the instructions to control the light in the way that it will regularly turn on and turn off. Afterwards, students try to use two input senders; they understand the sender's principle through the example of door opening and closing (they press one sender – the door opens, student enters and steps on the second sender – the door closes). Then, students work on the house map, which is equipped with the sensing devices (sender for controling the warmth, operation of the door, light and also, automatic monitoring of the house).

Internet as a information source and the interpretation of information 6D

Students learn how to use Internet as a source of information, they evaluate them and compare the veracity with the others.

Students learn to pick up useful information (from books, CD encyclopaedia or from the Internet), they inform classmates about the information. They create their own list of favourite links, they learn to find the information and print them. They use "key"words to find specific information and use the conjunction "and". They also learn to understand the copyright – even when they use pictures or text for their own presentations.

4 Conclusion

When we have studied the British curriculum, we must to table the question about its advantages and possible usage on our curriculum.

Furst of all, I would like to think about its conception, then I will compare it with our, which is included in “Framework curriculum for elementary education” (Rámcový vzdělávací program pro základní vzdělávání).

In our educational system, ICT came into the primary school with “RVP ZV”. ICT is, after several years of experience, part of the curriculum from the 4th of 5th grade of the primary school. Teachers should not be afraid of that ICT would be taught to students of younger school age, these students have already experience from their homes, where they do have PC and use it almost every day-eg. for playing video games. There I can see the importance of teaching children ICT.

When comparing both documents, the British standards are very precise, divided into separate chapters, unlike our Framework curriculum. Every chapter has its goal of what students should acquire. Then, series of tasks follow, which shoul help the teacher while working with students. There are always several outputs expected in all chapters that students–either slow or talented students–are expected to gain. The technical dictionary with an important key terms is included; there are links to important information sources too. There is a possibility to print the materials as well. The Czech Framework curriculim gives the teacher free hand, which could be problem for the compatibility at several different elementary schools; and also problem for the teacher, who would have difficulties to fulfil his aims during the lessons. As far as I am concerned, teaching ICT from the 4th grade certifies this fact. The ideal combination would be our Framework curriculum as it is with the addition to separate topics; but this Framework needs to be carefully formulated, as the British standards.

British standards show teachers the way of natural and spontaneous form of mathematical education in such way children get in the Great Britain.

I wanted to present this report as the inspiration of teaching the ICT to students of young school age.

References

[1] ICT at key stages 1 and 2. Available on http://www.standards.dfes.gov.uk/schemes2/it/?version=2

[2] *Rámcový vzdělávací program pro základní vzdělávání (RVP ZV).* Praha: VUP, 2005.

[3] PRŮCHA, J.; WALTEROVÁ E.; MAREŠ J. *Pedagogický slovník.* Praha: Portál, 2003. ISBN 80-7178-772-8.

[4] Kolektiv autorů. *Vzdělávání pro život v informační společnosti.* Praha: Univerzita Karlova v Praze, Pedagogická fakulta, 2005. ISBN 80-7290-202-4.

Random Variable as a Network of Pipelines

Roman Frič[1], Martin Papčo[2]

[1] Mathematical Institute
Slovak Academy of Sciences

Catholic University in Ružomberok,
Department of Mathematics, Faculty of Education

[2] Catholic University in Ružomberok,
Department of Mathematics, Faculty of Education

Abstract. In teaching probability, understanding the notion of a probability space and the ability to construct probability spaces modeling basic stochastic phenomena is one of the prime goals. In our talk we start with a known probability space modeling the tossing of two dice and the related probability space modeling the resulting sum of respective outcomes. The relationship between the two probability spaces has important historical aspects and represents an important paradigm. The transition from the first to the second can be described in terms of a random variable. We restrict ourselves to random variables for discrete probability spaces. We present a scheme which should lead to a better understanding of the rather technical notion of a random variable, to understand its importance, but also its limitations and the need of its generalization, i.e., the need of a transition from classical random variables to fuzzy random variables. Finally, we describe the quantum aspects of fuzzy random variables.

Keywords. Random variable, probability space, transformation, distribution pipeline, fuzzy event, fuzzy random variable, fuzzy probability.

1 Motivation

By a probability space we understand a pair (Ω, p), where $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ is a finite set and p is a map of Ω into $[0,1]$ such that each $p(\omega_l)$, $l = 1, 2, \dots, n$, is

[1]This research is supported by VEGA 1/0539/08, VEGA 2/0032/09.
[2]This research is supported by VEGA 1/0539/08.

nonnegative and $\sum_{l=1}^{n} \omega_l = 1$. If all $p(\omega_l)$ are positive, then (Ω, p) is said to be non-trivial (see [5]). The subsets of Ω model random events, the usual set operations model Boolean logic operations with random events, and the probability of a random event is usually interpreted as the chance that a given event occurs in the modeled random experiment. The probability $P(A)$ of an event $A \subseteq \Omega$ is calculated as $\sum_{\omega_l \in A} p(\omega_l)$. When modeling more sophisticated random experiments, also infinite sets Ω are considered, the random events are modeled by σ-fields of subsets, and the probability is modeled by a σ-additive normed measure. We will consider only finite spaces.

Throwing two dice, say D_1 and D_2, the outcomes can be modeled by ordered pairs $\omega = (x, y)$, where x is the number of points on D_1 and y is the number of points on D_2, i.e. $\Omega = \{(x, y),\ 1 \le x \le 6,\ 1 \le y \le 6\}$, and all pairs are equally probable, i.e. a map p is given by formula $p\big((x, y)\big) = \frac{1}{36}$. Thus the probability space (Ω, p) is the model of two dice experiment.

Let us model the sum of numbers of points on both die in previous random experiment. The possible sum $x + y$, $1 \le x \le 6$, $1 \le y \le 6$, varies from 2 to 12, i.e. $\Xi = \{2, 3, \dots, 11, 12\}$, and the probability $q(\xi)$ of a given sum ξ is the ratio $\frac{k}{36}$, where k is the number of all ordered pairs (x, y) such that $x + y = \xi$. Note that, for example, the probability of $x + y = 3$ is equal to the probability of the event $\{(1, 2), (2, 1)\}$, i.e., $\frac{2}{36}$. The probability space (Ξ, q) is another model of two dice experiment.

Formally, we have two probability spaces (Ω, p), (Ξ, q), the first consists of 36 elements, the second consists of 11 points, and the second space is a sort of quotient of the first one, where the quotient map T sends $\omega = (x, y)$ to $\xi = x + y$ and the probability of $\xi = x + y$ is equal to the probability of its preimage $T^{\leftarrow}(\xi) \subseteq \Omega$.

ATTENTION: This quotient type construction to get a new probability space from the original one is a paradigm leading to the notion of a random variable. Indeed, the quotient map is a channel through which a probability on the original space can be moved to a probability on the new space. In fact, more important structural role is played by the preimage map (dual to the quotient map): it is a Boolean homomorphism mapping the random events of the new space into the random events of the original space. It is called observable and plays a similar role as a linear map does in the theory of linear spaces, or a group homomorphism plays in group theory. Our first goal is to describe a random variable as a network of pipelines through which the probability on the original space can be moved to a probability on the new space.

Definition 1. Let (Ω, p) and (Ξ, q) be probability spaces. Let T be a map of $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ into $\Xi = \{\xi_1, \xi_2, \dots, \xi_m\}$ such that $q(\xi_k) = \sum_{\omega_l \in T^{\leftarrow}(\xi_k)} p(\omega_l)$ for all $k \in \{1, 2, \dots, m\}$ such that $q(\xi_k) > 0$. Then T is said to be a transformation of (Ω, p) to (Ξ, q) and (Ξ, q) is said to be the T-image of (Ω, p). If Ξ is a set of real numbers, then T is said to be a random variable.

Let T be a transformation of a probability space (Ω, p), $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$, to a probability space (Ξ, q), $\Xi = \{\xi_1, \xi_2, \dots, \xi_m\}$. Then T can be visualized as a system of n pipelines $\omega_l \mapsto T(\omega_l)$ through which $p(\omega_l)$ flows to $\xi_k = T(\omega_l)$. If ξ_k is the target of several pipelines, then $q(\xi_k)$ is the sum $\sum_{\omega_l \in T^{\leftarrow}(\xi_k)} p(\omega_l)$, i.e., the total influx through the pipelines in question. (See Figure 1.)

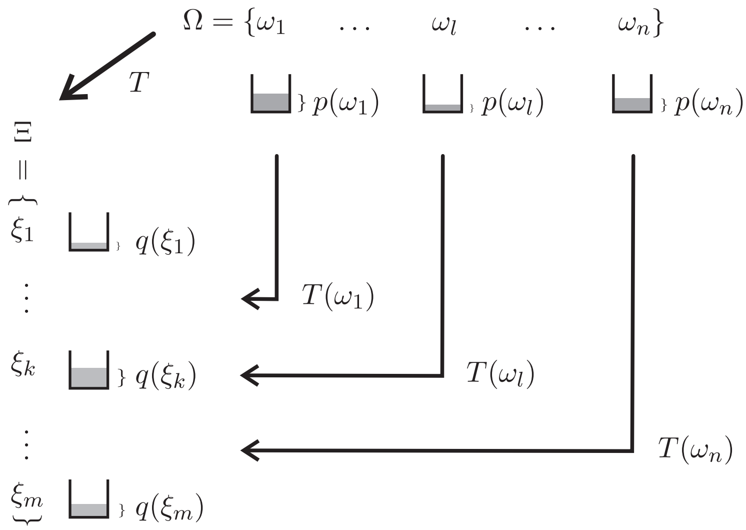


Figure 1.

Let (Ω, p) be a probability space. Clearly, the identity map on Ω is a "trivial" transformation of (Ω, p) to itself. Observe that there are also nontrivial transformations (e.g. consider the tossing of a regular coin and the reversing of head and tail).

Note that, for discrete probability spaces, random variables are special transformations, where the underlying set of the target probability space is a set of real numbers.

Let (Ω, p) and (Ξ, q) be probability spaces.

QUESTION: Does there always exist a transformation of (Ω, p) to (Ξ, q)?

ANSWER: NO.

Indeed, it is easy to see that for nontrivial probability spaces (Ω, p) and (Ξ, q), if Ξ has more points than Ω, then there is no transformation of (Ω, p) to (Ξ, q).

2 Est modus in rebus

However, "est modus in rebus": instead of sending each $p(\omega_l)$ to some ξ_k via a simple "pipeline" $\omega_l \mapsto \xi_k = T(\omega_l)$, we can try to distribute $p(\omega_l)$, simultaneously sending to each ξ_k, $k \in \{1, 2, \ldots, m\}$, via a complex "distribution pipeline" some fraction $w_{kl}p(\omega_l)$ of $p(\omega_l)$. Of course, not arbitrarily, but in such a way that the fractions sum up "properly", i.e., $\sum_{l=1}^{n} w_{kl}p(\omega_l) = q(\xi_k)$ and $\sum_{k=1}^{m}\sum_{l=1}^{n} w_{kl}p(\omega_l) = \sum_{l=1}^{n} p(\omega_l)\sum_{k=1}^{m} w_{kl} = \sum_{k=1}^{m} q(\xi_k) = 1$. To comply with the second condition it suffices to guarantee that $\sum_{k=1}^{m} w_{kl} = 1$. In fact, this means that to each ω_l,

$l \in \{1, 2, \ldots, n\}$, we assign a suitable probability function $q_l = (w_{l1}, w_{l2}, \ldots, w_{lm})$ on Ξ. (See Figure 2.)

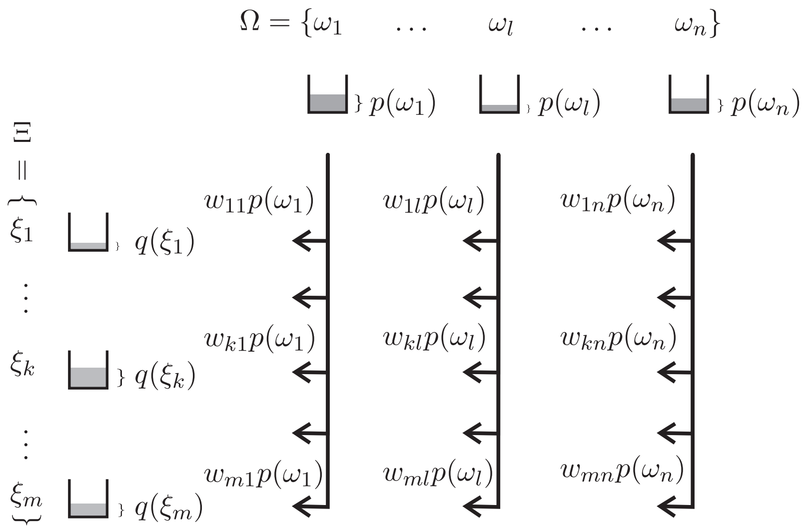


Figure 2.

The construction of a "distribution pipeline" yields a generalized transformation of (Ω, p) to (Ξ, q); p flows trough the pipeline and it is transformed into q. The generalized transformation has a surprising background: fuzzy probability.

Let (Ω, p), $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, and (Ξ, q), $\Xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$, be probability spaces. Let $\{r_{kl};\, l \in \{1, 2, \ldots, n\},\, k \in \{1, 2, \ldots, m\}\}$ be nonnegative numbers such that $\sum_{k=1}^{m} r_{kl} = p(\omega_l)$, $l \in \{1, 2, \ldots, n\}$ and $\sum_{l=1}^{n} r_{kl} = q(\xi_k)$, $k \in \{1, 2, \ldots, m\}$. For $l \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$ define $w_{kl} = 1/m$ if $p(\omega_l) = 0$ (any choice such that $\sum_{k=1}^{m} w_{kl} = 1$ does the same trick) and $w_{kl} = \frac{r_{kl}}{p(\omega_l)}$ otherwise. Clearly, $\sum_{l=1}^{n} w_{kl} p(\omega_l) = q(\xi_k)$ for all $k \in \{1, 2, \ldots, m\}$ and $\sum_{k=1}^{m} w_{kl} = 1$. This yields a "distribution pipeline".

Observations.

1. The "distribution pipeline" is a matrix W consisting of m rows $w_k = (w_{k1}, w_{k2}, \ldots, w_{kn})$ and n columns $q_l = (w_{1l}, w_{2l}, \ldots, w_{ml})$, where $w_{kl} \in [0, 1]$ and each q_l is a suitable probability function on Ξ.

2. Each q_l distributes the content $p(\omega_l)$ among the points $\xi_k \in \Xi$ (respecting $\sum_{l=1}^{n} w_{kl} p(\omega_l) = q(\xi_k)$).

3. Each w_k is a map of Ω into $[0, 1]$, i.e. a fuzzy set, and w_{kl} determines how much of $p(\omega_l)$ flows to ξ_k.

3 Interpretations

1. **Conditional probability**. Observe that $r = \big\{r_{kl};\, l \in \{1,2,\ldots,n\},\, k \in \{1,2,\ldots,m\}\big\}$ yields a probability space $(\Omega\times\Xi, r)$ on the product set $\Omega\times\Xi$ such that (Ω, p) and (Ξ, q) are marginal probability spaces. Thus r and, consequently, the "distribution pipeline" is not uniquely determined.

Now, assume that $p(\omega_l)$ is positive. Then each w_{kl} can be interpreted in terms of the conditional probability in the probability space $(\Omega \times \Xi, r)$. Indeed, for $l \in \{1,2,\ldots,n\}$ put $A_l = \{(\omega_l,\xi_1),(\omega_l,\xi_2),\ldots,(\omega_l,\xi_m)\}$ and for $k \in \{1,2,\ldots,m\}$ put $B_k = \{(\omega_1,\xi_k),(\omega_2,\xi_k),\ldots,(\omega_n,\xi_k)\}$. Then $w_{kl} = \frac{r_{kl}}{p(\omega_l)} = \frac{P(B_k\cap A_l)}{P(A_l)}$, which is the same as $P(B_k \mid A_l)$, the conditional probability of B_k given A_l. Clearly, if $r_{kl} = q(\xi_k)p(\omega_l)$, then B_k and A_l are independent.

2. **Quantum character**. Unlike in the classical probability, where a transformation (a random variable) sends each point to a point (a number), a "distribution pipeline" can send a point $\omega_l \in \Omega$ to a (genuine) probability function q_l. This has definitely a quantum character.

3. **Fuzzy character**. To each crisp event $\{\xi_k\}$ there corresponds a fuzzy set $w_k = (w_{1k}, w_{2k},\ldots,w_{nk})$ determining how much of p flows to ξ_k; w_k is called a *fuzzy event* and the fuzzy p-probability of w_k is equal to the q-probability of ξ_k. So, the quantum character of a "distribution pipeline", leads to fuzzy events, fuzzy probability and fuzzy observables (sending crisp events to fuzzy events). Of course, if for a given k all w_{kl} are either 1 or 0, then w_k collapses to the crisp event containing all ω_l such that the whole $p(\omega_l)$ flows to ξ_k. If for each l, $l \in \{1,2,\ldots,n\}$, some $w_{kl} = 1$, then the complex "distribution pipeline" collapses to n simple pipelines and the generalized (fuzzy) transformation of (Ω, p) to (Ξ, q) collapses to a classical one.

4 Concluding remarks

Quantum and fuzzy probability provide fundamental tools to model stochastic situations which the classical probability cannot deal with. We believe that the idea of a pipeline and a distribution pipeline will help to understand basic probability notions and will lead to interesting applications.

Even though quantum phenomena are usually connected to subatomic scales, the idea of a distribution pipeline shows that generalized probability covers also situations from everyday life.

An interested reader can find more information about fuzzy probability for example in [1], [2], [3], [4].

References

[1] GUDDER, S. Fuzzy probability theory. *Demonstratio Math.*, 1998, 31, 235–254.

[2] FRIČ, R. Nonelementary notes on elementary events. *Pr. Nauk. Akad. Jana Długosza Czest. Mat.*, 2003, IX., 137–141.

[3] FRIČ, R. Remarks on statistical maps and fuzzy (operational) random variables. *Tatra Mt. Math. Publ.*, 2005, 30, 21–34.

[4] PAPČO, M. On fuzzy random variables: examples and generalizations. *Tatra Mt. Math. Publ.*, 2005, 30, 175–185.

[5] PŁOCKI, A. *Pravdepodobnosť okolo nás.* Ružomberok: Katolícka univerzita v Ružomberku, 2007.

An Investigation Involving Positive Integers

Jan Kopka[1], George Feissner[2], Leonard Frobisher[3]

[1] Jan Evangelista Purkyně University in Ústí nad Labem, Faculty of Science, Department of Mathematics

[2] State University of New York, College at Cortand, Department of Mathematics

[3] 58 Buckstone Avenue, Leeds, LS 17 5HP, England

Abstract. Here we demonstrate investigative approach and creation of conjectures with the help of a problem.A conjecture is a statement which appears reasonable, but whose truth has not yet been established. Conjecturing forms the backbone of mathematical thinking and reasoning.

Keywords. Investigation, integer, experimentation, conjecture, test, proof, mathematical theorem.

We present an approach to investigate a “mathematical situation”. We choose not to define “mathematical situation” but will illustrate it with several examples. Since it is the first step on what can be a long and interesting journey, we feel it is best to allow it to be as general as possible.

A typical method of investigating a mathematical situation follows this sequence of processes:

Mathematical situation – experimentation – conjecture – test – proof – mathematical theorem.

The central activity during the investigation of many mathematical problem situations is that of conjecturing. A conjecture is a statement which appears reasonable, but whose truth has not yet been established. Conjecturing forms the backbone of mathematical thinking and reasoning. We look at a number of famous and interesting conjectures associated with positive integers which can be explored at different levels in elementary and secondary schools.

Example 1 (Goldbach's Conjecture).

Many millions of even numbers have been examined and every one tested can be written as the sum of two prime numbers. Some can be written this way in several different forms. For instance,

$$\begin{array}{ll} 4 = 2 + 2 & 10 = 5 + 5 = 3 + 7 \\ 6 = 3 + 3 & 12 = 5 + 7 \\ 8 = 3 + 5 & 14 = 7 + 7 = 3 + 11. \end{array}$$

But as it is impossible to test every even number somewhere there may be at least one even number that cannot be written as the sum of two prime numbers. So in mathematics there is the very famous Goldbach's **conjecture**:

Every even number greater than 2 *is the sum of two prime numbers.*

We exclude 2 since the only way to write 2 as the sum of two positive integers is 1 + 1 and we do not consider 1 a prime number.

But as yet no one has found a proof that every even number has this property, and it may not even be true. So the statement is "only" a conjecture, not a mathematical theorem. In the future some mathematician may prove the conjecture to be true, but equally some other mathematician may prove it not to be true.

The Prussian mathematician Christian Goldbach (1690 - 1764) made this conjecture in a conversation with the great Euler who later wrote Goldbach to say that "I regard this as a completely certain theorem, although I cannot prove it." It remains unproven to this day and indeed may not be true!

Example 2 (Ulam's Conjecture).

This example involves producing finite sequences.

Choose any positive integer n.

1. If n is even, divide it by 2.

2. If n is odd, multiply it by 3, add 1, then divide it by 2.

3. If the new number is 1, stop; otherwise, repeat.

For example, if $n = 8$, we obtain the sequence $8 \to 4 \to 2 \to 1$.

If $n = 13$, we obtain $13 \to 20$ $((13 \cdot 3 + 1)/2) \to 10 \to 5 \to 8$ $((5 \cdot 3 + 1)/2) \to 4 \to 2 \to 1$.

Will we always get 1? We can experiment further and then state a conjecture:

Conjecture. *For any positive integer n, the above procedure will always terminate with* 1*.*

Although often called Ulam's Conjecture after the Polish-American mathematician Stanislaw Ulam (1909 - 1984), it was originally stated by the German mathematician Lothar Collatz (1910 - 1990). Paul Erdös remarked that "Mathematics is not yet ready for such problems." As with Goldbach's Conjecture, it is unproven.

It is not true that all conjectures lead to unsolved problems. Now we demonstrate investigative approach and creation of conjectures with the help of the folloving problem.

Problem: Sum of Consecutive Positive Integers

What positive integers can be written as a sum of consecutive positive integers? For instance, $7 = 3 + 4$ and $15 = 4 + 5 + 6$, but we cannot write 8 as such a sum (try it).

Solution

Because we do not know anything about it we must start by using the process of experimentation. There are at least two ways of doing this.

– The first way is some times known as the heuristic 'Working backwards'. Take sets of two, then three, then four consecutive numbers, and so on, and find their sums.

  For example:

$$\begin{array}{lll} 1+2=3 & 1+2+3=\ \ 6 & 1+2+3+4=10 \\ 2+3=5 & 2+3+4=\ \ 9 & 2+3+4+5=14 \\ 3+4=7 & 3+4+5=12 & 3+4+5+6=18 \\ \quad\ldots & \qquad\ldots & \qquad\qquad\ldots \end{array}$$

  This shows us that the integers 3, 5, 7, 6, 9, 12, 10, 14, 18, can be written as the sum of consecutive numbers. Let's order these numbers to see if we can see a pattern.

$$3, 5, 6, 7, 9, 10, 12, 14, 18 \ldots$$

If we continue this approach perhaps we might fill in the missing numbers in this sequence or notice a pattern in possible missing numbers. We leave you to continue this approach.

– The second way takes each integer in turn and tries to express it as a sum of consecutive numbers. Here we use the heuristic known as 'Working systematically'.

These are the results of experimenting with the integers 1, 2, 3, 4 and 5:

1 = 0 + 1	Not valid! We must use positive integers and 0 is not positive.
2 =	We cannot write 2 as the sum of consecutive positive integers.
3 = 1 + 2	Yes!
4 =	Again, we cannot find a sum.
5 = 2 + 3	Yes!

After experimenting with the first five integers it is possible to make a conjecture:

Conjecture 1. *Even numbers are not the sum of consecutive positive integers.*

Let us experiment further still working systematically:

6 = 1 + 2 + 3	Yes!
7 = 3 + 4	Yes!
8 =	We cannot find the sum.
9 = 2 + 3 + 4	Yes!
10 = 1 + 2 + 3 + 4	Yes!
11 = 5 + 6	Yes!

The experimentation with integers 6 to 11 shows that conjecture 1 is not true. For example, 6 and 10 are even integers, and 6 = 1+2+3 and 10 = 1+2+3+4 disprove it. But what about the integer 8? Is it possible to produce another conjecture knowing that the integers 1, 2, 4 and 8 cannot be written as the sum of consecutive integers? What is special about 1, 2, 4 and 8? We know them as powers of 2. Although we have only four of these special numbers we claim:

Conjecture 2. *Powers of two are cannot be written as the sum of consecutive positive integers.*

If we **test** conjecture 2 with the help of number $16 = 2^4$, we more confident that this conjecture is true. But at this moment we do not know how to construct a proof.

Now as we can continue our **experimentation** working systematically with more and more integers it is possible to say:

Conjecture 3. *All numbers with the exception of powers of two are sums of consecutive positive integers.*

How can we prove conjectures 2 and 3? We must find what distinguishes powers of two from other numbers and then show how this property relates to the property of being a sum ofconsecutive numbers.

From divisibility theory, we know that the only prime factor of a power of two is two itself. Thus, all factors of a power of two other than 1 are even (indeed, they are also powers of two). On the other hand, any positive integer not a power of two must have at least one odd factor other than 1.

Example

Factors of $2^5 = 32$ are: 32, 16, 8, 4, 2, 1. All factors except 1 are even.
Factors of 24 are: 24, 12, 6, 4, 3, 2, 1. There is an odd factor other than 1. It is number 3.

Now we have a distinguishing property, so we can reformulate our conjectures.

Conjecture 2 a. *If a positive integer has only even factors other than* 1*, then it cannot be written as the sum of consecutive positive integers.*

Conjecture 3 a. *If a positive integer has at least one odd factor other than* 1 *then it can be so written.*

Further experimentation can help us see how the presence of an odd factor allows us to write an integer as a sum of consecutive positive integers.

Experimentation: Let us take some numbers with at least one odd factor, say multiples of 3, 5 and 7, and examine them. Our experimentations show the following forms:

Multiples of 3:

$$
\begin{aligned}
1 \cdot 3 &= 1 + 2 \\
\mathbf{2} \cdot 3 &= 1 + \mathbf{2} + 3 \\
\mathbf{3} \cdot 3 &= \quad\; 2 + \mathbf{3} + 4 \\
\mathbf{4} \cdot 3 &= \qquad\;\; 3 + \mathbf{4} + 5
\end{aligned}
$$

In general:

$$g \cdot 3 = (g - 1) + g + (g + 1)$$

Multiples of 5:

$$
\begin{aligned}
1 \cdot 5 &= 2 + 3 \\
2 \cdot 5 &= 1 + 2 + 3 + 4 \\
\mathbf{3} \cdot 5 &= 1 + 2 + \mathbf{3} + 4 + 5 \\
\mathbf{4} \cdot 5 &= \quad 2 + 3 + \mathbf{4} + 5 + 6
\end{aligned}
$$

In general:

$$g \cdot 5 = (g - 2) + (g - 1) + g + (g + 1) + (g + 2)$$

Multiples of 7:

$$
\begin{aligned}
1 \cdot 7 &= 3 + 4 \\
2 \cdot 7 &= 2 + 3 + 4 + 5 \\
3 \cdot 7 &= 1 + 2 + 3 + 4 + 5 + 6 \\
\mathbf{4} \cdot 7 &= 1 + 2 + 3 + \mathbf{4} + 5 + 6 + 7 \\
\mathbf{5} \cdot 7 &= \quad 2 + 3 + 4 + \mathbf{5} + 6 + 7 + 8 \\
\mathbf{6} \cdot 7 &= \qquad 3 + 4 + 5 + \mathbf{6} + 7 + 8 + 9
\end{aligned}
$$

In general:

$$g \cdot 7 = (g - 3) + (g - 2) + (g - 1) + g + (g + 1) + (g + 2) + (g + 3)$$

Note that these general formulas only hold when g is sufficiently large. In the above examples, the cases where the formula holds true are those where some of the numbers appear in bold-face.

And now generally: If positive integer n has an odd factor $2k + 1$ (k is positive integer), it means that n can be written in the form

$$n = g(2k + 1). \tag{1}$$

Then n can also be written as a sum

$$n = (g - k) + \cdots + (g - 2) + (g - 1) + g + (g + 1) + (g + 2) + \cdots + (g + k). \tag{2}$$

Really, if we add the terms on the right hand side of formula (2), we get the term on the right hand side of formula (1).

Formula (2) says that such an n is the sum of $2k+1$ consecutive integers, the middle one being g. This is almost Conjecture 3 a, but unfortunately, not all the integers need be positive! For instance, if we apply the formula to $n = 3, 5, 7, 10,$ and 14, we obtain

$$
\begin{aligned}
3 = 1 \cdot 3 &= \mathbf{0} + 1 + 2 \\
5 = 1 \cdot 5 &= \mathbf{-1} + \mathbf{0} + 1 + 2 + 3 \\
7 = 1 \cdot 7 &= \mathbf{-2} + (\mathbf{-1}) + \mathbf{0} + 1 + 2 + 3 + 4 \\
10 = 2 \cdot 5 &= \mathbf{0} + 1 + 2 + 3 + 4 \\
14 = 2 \cdot 7 &= \mathbf{-1} + \mathbf{0} + 1 + 2 + 3 + 4 + 5.
\end{aligned}
$$

However, this last experiment has shown us exactly what we need to do to show that Conjecture 3 a is true. We present it as a proof.

Proof of conjecture 3 a: Let positive integer number n have an odd factor $2k+1$. Then n can be written in the form (1). Now we know, that n can be written also in the form (2)

$$n = (g-k) + \cdots + (g-2) + (g-1) + g + (g+1) + (g+2) + \cdots + (g+k) \qquad (2)$$

where g is also factor of n.

How can we get only two or more positive numbers on the right hand side of formula (2)? If there is zero, we delete it. All we have to do is counteract the negative terms by the corresponding positive terms. There are more positive terms than negative ones because the total sum is positive. After this procedure we get the even number of consecutive positive terms.

So after this proof we rename conjecture 3 a as a theorem:

Theorem 1. *If a positive integer n has an odd factor other than* 1*, then it can be written as the sum of consecutive positive integers.*

Indeed, our investigation has given us more than what the theorem states; not only do we know that it can be done, but we know **how** to do it!

Now we have enough experiences to believe that the opposite theorem to theorem 1 also holds.

Theorem 2. *If a positive integer n can be written as the sum of consecutive positive integers, then it must have an odd factor other than* 1*.*

Proof: We write n as $n = g_1 + g_2 + \cdots + g_r$ where $g_1, g_2, \dots, g_r$ are consecutive positive integers. We consider two cases:

(a) r is an odd number

(b) r is an even number.

Case (a) We can write n in the form shown in Formula (2), and then g will be the number in the middle of the sum. The fact that r is odd guarantees there is a middle term. Then, exactly as in our experimentation, we can then write n in the form $n = g(2k+1)$ so n has an odd factor greater than 1, namely $2k+1$.

Case (b) We are unable to proceed as in Case (a) since now there is no middle term. However, we "augment" the sum $g_1 + g_2 + \cdots + g_r$ by adding to it

$$-(g_1-1) + \cdots + (-2) + (-1) + 0 + 1 + 2 + \cdots + (g_1-1).$$

> Note first that we have really just added on 0, since all the positive terms cancel with the corresponding negative ones. Thus, the new sum is still equal to n. Second, we have added on an odd number of terms – an equal number of positive and negative terms and 0. Since we began with an even number of terms in the sum $g_1 + g_2 + \cdots + g_r$ we now have an odd number of terms in the augmented sum, namely an odd number of consecutive integers whose sum is n. The situation is now exactly as it was in Case (a), so we can conclude that n has an odd factor greater than 1.

It is difficult to say exactly how this "trick" of augmenting the sum in Case (b) was conceived. That is part of the creative side of mathematics. However, we can say with confidence that had we not been thinking of sums of consecutive integers and of Formula (2), the trick would not have been thought of.

We now put both theorems together:

Theorem 3. *A positive integer n can be written as the sum of consecutive positive integers if and only if the positive integer n has an odd factor other than one.*

Corollary of theorem 3 is conjecture 2. The powers of two have no odd factors except 1, so these powers cannot be expressed as the sum of consecutive positive integers. Therefore we can rename conjecture 2 as theorem 4:

Theorem 4. *Powers of two are not the sum of consecutive positive integers.*

Now we are inspired for another investigation.

Some positive integers can be written as a sum of consecutive positive integers in different ways. For example:

$$21 = 3 \cdot 7 \qquad 21 = (0+)1 + 2 + \mathbf{3} + 4 + 5 + 6$$
$$= 6 + \mathbf{7} + 8$$
$$= 10 + 11$$
$$30 = 2 \cdot 3 \cdot 5 \qquad 30 = 9 + \mathbf{10} + 11$$
$$= 4 + 5 + \mathbf{6} + 7 + 8$$
$$= 6 + 7 + 8 + 9$$

We pose a new problem.

Problem: Investigate the number of different ways a given positive integer can be written the sum of consecutive positive integers.

Here is our **investigation**:

Again we can start with the process of **experimentation**. We can also **use** our **mini-theory**, which was created above. But first several specific questions.

Question:	Which numbers have no representation as the sum of consecutive integers?
Answer:	Powers of 2.
Question:	Which numbers have a unique representation?
Answer:	Those with only one odd factor other than 1, namely primes and products of a power of 2 and a prime, such as $56 = 8 \cdot 7 = 2^3 \cdot 7$.
Question:	Which numbers have exactly two such representations?
Answer:	Those with exactly two distinct odd factors other than 1, such as $9 = 3^2$. The two distinct factors are 3 and 9.
Question:	Which numbers have exactly three such representations?
Answer:	Those with exactly three distinct odd factors other than 1, such as $42 = 2 \cdot 3 \cdot 7$. The three distinct odd factors are 3, 7, and 21.

We are now in a position to offer a further conjecture.

Conjecture 4. *Let n be positive integer. The number of representations of n as a consecutive sum is the same as the number of odd factors of the integer n.*

We are pretty certain that the conjecture is true, but can we prove it?

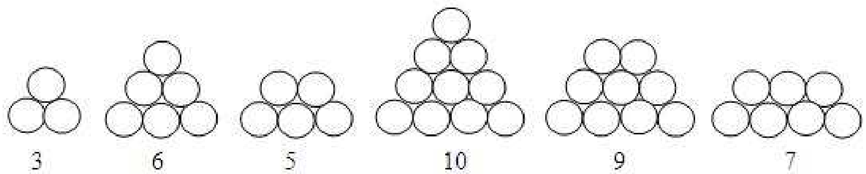
Proof: For every odd factor $2k+1$ of n we can construct sum (2) and after counteract we always get a sum of consecutive positive integers.

For example: Number $30 = 2 \cdot 3 \cdot 5$ has three odd factors: 3, 5 and $3 \cdot 5 = 15$.

Sum of 3 consecutive integers: $30 = 9 + 10 + 11$
Sum of 5 consecutive integers: $30 = 4 + 5 + 6 + 7 + 8$
Sum of 15 consecutive integers (or 4 consecutive positive integers):
$30 = (-5)+(-4)+(-3)+(-2)+(-1)+0+1+2+3+4+5+6+7+8+9 = 6+7+8+9$

The conjecture 4 can now be claimed to be a **theorem 5**.
We **end** with two remarks of a didactic nature:

Remark 1. We can use a geometric approach to introduce younger students to this problem with the help of figures such as these:



Diagrammatising numbers in geometric shapes helps students visualise patterns in particular numbers. The shapes could be said to represent an assembly of pipes or logs of wood. The numbers below the figures show the number of pipes in a shape and we can refer to numbers as 'pipe' numbers. So we investigate which numbers are pipe numbers. Younger students can also be introduced to experimentation in mathematics by supplying them with a collection of metal or plastic disks and having them "construct" pipe numbers.

A more algebraic view:

It is easy to derive the formula $1+2+3+\cdots+n = n(n+1)/2$. There are both algebraic and geometric ways to do so. Now, a "sum" number such as $5 + 6 + 7 + 8 + 9 + 10$ can be looked at as

$$(1 + 2 + 3 + \cdots + 9 + 10) - (1 + 2 + 3 + 4) = (10 \cdot 11)/2 - (4 \cdot 5)/2.$$

Thus, a "sum" number is one that can be written either in the form $n(n + 1)/2$ or $n(n + 1)/2 - m(m + 1)/2$ where in the second case, $m + 1 < n$.

See how this relates to the figures: the first formula (where $n = 2, 3, 4, 5, \ldots$) produces triangles, more precisely, the triangular numbers: $3, 6, 10, 15, \ldots$ The second formula (where $n = 2, 3, 4, 5 \ldots$, $m = 1, 2, 3, 4, \ldots$ and $m + 1 < n$) produces truncated triangles, or more precisely, the difference of two triangular numbers. If in the second formula we allow $m = 0$, then this formula produces all the "sum" numbers.

Remark 2. Students just beginning algebra can be led to the important formula

$$n = (g - k) + \cdots + (g - 2) + (g - 1) + g + (g + 1) + (g + 2) + \cdots + (g + k)$$

by considering examples such as

$$4 \cdot 3 = 4 + 4 + 4 = (4 - 1) + 4 + (4 + 1)$$

or

$$6 \cdot 5 = 6 + 6 + 6 + 6 + 6 = (6 - 2) + (6 - 1) + 6 + (6 - 1) + (6 - 2).$$

In each such sum, it is important to begin with an odd number of copies of a given integer so there will be a middle term. Then as we move away from the middle term to the left, we subtract off 1, then 2, then 3, and so on, while we add these numbers when we move from the middle to the right. It is important that the student sees that doing so does not change the sum.

References

[1] KOPKA, J. *Výzkumný přístup při výuce matematiky.* Ústí nad Labem: Univerzita Jana Evangelisty Purkyně, 2007. ISBN 978-80-7044-926-4.

[2] KOPKA, J. Isolated and Non-isolated Problems. *Teaching Mathematics and its Applications.* Oxford University Press, 1997, vol. 16, no. 1, p. 27–35. ISSN 0268-367.

[3] MASON, L. *Thinking Mathematically.* New York: Addison-Wesley Publishing Copany, 1994.

[4] ORTON, A.; FROBISHER, L. *Insights into Teaching Mathematics.* New York: Cassell, 1996.

DEVELOPMENT OF MATHEMATICAL TERMS IN PRELIMINARY CLASSES

Jan Melichar[1]

Jan Evangelista Purkyně University in Ústí nad Labem,
Faculty of Education, Department of Mathematics and ICT

Abstract. The article briefly describes history and sense of preliminary classes. It shows coherence with educational framing programs, emphasizes the process of terms formation and development of mathematical thinking in preliminary classes.

Keywords. Preliminary classes, educational framing programs, term, mathematical thinking.

1 Introduction - preliminary classes for children from social-cultural handicaped environment

Since 1993 has grown amount of children that have not attended kindertgarten and have not suitable family background for their positive development. It concerned mainly children from various ethnic groups or immigrants. Primarily immigrant-parents had to solve integration to Czech society. Since 1993 has been solved this problem by establishing of preliminary classes (at that time called "zero classes") at basic schools. Those classes were established without statutes. On 1st September 1999 came in force "Statutes of Experimental Follow-up of Preliminary Classes For Children From Social-cultural Handicaped Environment" appointed by the of Education of Czech Republic. The Statutes set down i.a. that school attendance of preliminary classes is not compulsory. School timetable was not settled. Teachers generated daily activities by themselves based on needs, skills, habits and actual

[1]The article is part of the ESF Project "We Can Do It Together – Programm of Pedagogical Intervention for Children from Social-cultural Handicaped Environment and Their Teachers", reg. No. of the project: CZ.1.07/1.2.00/08.0105.

attention of children. Game is the main form of work. Verbal communication is developed e.g. by the means of dramatic education. Preliminary classes help children from ethnic minorities to integrate successfully into the process of education. Preliminary classes are connecting links between kindergarten and first class of primary school. The aim is a systematical preparation of children for integration into the process of education in the first class of basic school. The content of education in preliminary classes does not contain themes that are taught in first class of basic school, instead of that are used alternative activities.

Even if parents put an emmense effort to education of their children they can hardly make up children's society, competition, assimilation, adaptation, accustom to teacher and other people.

In 2004 was passed the **School Law (No. 561/2004 about pre-school, basic, secondary and other education)** that empowers schools to establish preliminary classes.

2 Educational Program - Development of Mathematical Thinking

2.1 Development of Mathematical Thinking

Development of mathematical thinking is an inevitable part of education in preliminary classes.

It is obvious, that during development of mathematical thinking we can not expect from children in preliminary classes knowledge of mathematical terms, theories, mathematical terminology, phraseology and symbology. We are only beginning the process of mathematical thinking. It results from the **Educational Framing Program for Pre-school Education** - part **Children and Its Psyche**, where we pay attention to the area **Language and Speaking**. For development of mathematical thinking is important to concentrate on development of speaking abilities and receptive language capabilities (perception, comprehension, listening) and productive language capabilities (pronunciation, formation of terms, speech and phrases). In the area **Cognitive Faculty and Function, Thought Operations, Imagination and Fantasy** we concentrate on child's full perception by all its senses, conscious concentration on work and attention keeping, identification of mathematical objects, concentration on things that are important form the cognitive point of view. Child should determine important marks, features of subjects, define characteristic marks of objects and events, connections between them. Child should act and learn according to given instructions, understand basic numeric and mathematical terms, elemental mathematical relations and use them adequately (compare, classify and sort sets of objects according to given rule, apprehend elemental computation up to six, series numerical up to ten, understand terms: more, less, first, last etc. Child should recognise spatial terms: left, right, up, down, in the middle, under, bellow, in etc. in space and plane. Child should partially understand time terms.

2.2 Thinking and Language

Thinking and Language are complemetary connected phenomena when thinking as the highest form of reality presentation is expressed and realised by the means of language. Thinking is connected with language, physiologicaly is thinking and language determined by second signal system and is used for world recognition and communication between people. Language represents thinking and it is its physiological determiner.

The oldest known definition of a sentence is from ancient times "*Oratio est ordinatio dictionum sententiam perfectam demonstrans*", it means "**Sentence** is coherent formation of words presenting completed idea".

2.3 Process of Word Formation

Let's look theoretically at two basic pillars of the process of education in the area of mathematical thinking. The first pillar is the process of word formation and the second one is the process of term selection.

Term is one of the forms of scientific cognition represented in our conscious and lately in our thinking significant marks of surveyed objects and relations.

In mathematics is term very often determined not only by term (word or group of words) – name but also by a symbol.

Terms help us to understand each other and to have similar idea about written or pronounced word. When you say the word "square" we have in our mind picture of planar pattern/figure delimited by four identical abscissae with inner angles 90°. However, each of us imagine square of different size and probably even colour. Anyway, significant features (marks) that determine the square are the same. If I say e.g. "young woman" we agree on a woman of young age but the idea of her appearance will differ. This term is very vague and based on its features can not be clearly identified. In mathematic we set such features (marks) in order to identify the term clearly, precisely.

Each term has certain content and range.

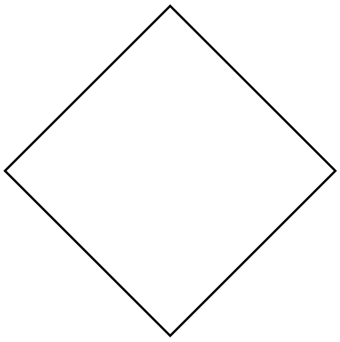
Content of a term is formed by a group (set) of all features (marks) that are characteristic for this term.

Range of a term is formed by a set of all objects that have features (marks) characterized by its content.

Children are learning terms, including mathematical ones, in terms. The more children are learning about the content and range of terms the more are terms getting clearer to them. E.g. we are showing to a child various rectangles together with different geometrical figures. The child forms in its conscious certain idea about the content of the term "rectangle". During the first term forming we are not decribing basic features (marks) of the given term. We are forcing the child to use as much senses as possible during the observation. Child is using sight it is called **visualization**. The term **visualization** includes the ability of visual perception and remembering of the seen object after a time. Child uses hearing together with visual perception during the presentation of figures. It is recommended to use also touch.

The rectangle can be made from carton or plastic material and child is trying to identify it among other objects with closed eyes. It is said that known things/ken is what was recognised by our senses. When child is older we are pointing out features (marks) of given terms. **Intuitive** cognition - when child forms in its mind content and range of a term.

We recquire form child to draw rectangle, to show rectangular objects, in a group of given models, identify object that has features of rectangle. Concerning geometric terms, it is problem that we operate with abstract terms that in reality do not exist. Only their models are real. Rectangle is a part of plane that has no “thickness”. That is the reason why it is difficult to form terms in geometry.



We should beware of “delusion”. Many people, even well-educated, will declare this figure e.g. traffic sign major road, is a rhomb. But it is not a rhomb, it is a square because it has all features (marks) of square.

It is a mistake to define a term to children. It is not important in the first part of the cognitive process. E.g. term quadratics - teacher in the first class of basic school was showing to children various records and was telling them “This is quadratics”, “This is not quadratics”. Children were recognizing quadratics among records. The teacher defined quadratics as a “record that contains a letter ‘x’ ”. We were present at this lesson so we asked one child to present a word “saxophone” to be a quadratics. The teacher’s definition was impeached. It is difficult to define terms e.g try to to define “table”.

While forming term it is important people have in their imagination as many identical marks as possible when we are showing them the symbol or pronouncing name of the symbol. In our society we can see misunderstanding regarding terms like democracy, freedom, privatisation etc. The aim is to offer such marks and as many marks as possible in order to define a term the most precisely. Personal experience is very important during the term-forming process e.g. literate person has different idea of pencil than illiterate person. Literate person considers pencil to be a writing tool, illiterate person see in pencil tool for piercing. It is important to define features of a term and regulate children’s experience.

2.4 Term classification

We will concentrate on theory of **classification**.

Classification of mathematical terms

The content of a term we define by means of definitions, the range by means of classification.

Elements with the same characteristic basic features (marks) belonging to a range of a given term form a set. Elements of the set can differ in supplementary marks or in quality or quantity of charakteristic feature (mark). During classification we proceed fragmentation of a given set (range of term) to classes (subsets) according to supplementary features (marks).

Classification must meet following criteria:

1. Classification has to be **complete** - it has to cover all elements of the set (range of term).

2. Classification has to be **disjoint** - it means that each element of classified set belongs to one class, no element can belong to two classses.

3. Classification has to be done based on **same mark** (feature).

There are very often made mistakes in classification e.g. when we ask to classify triangles, the respond is: triangles are acute, impedance, isosceles and equilateral. The third condition – classification based on same mark (feature) – was broken.

Complete classification of elements belonging to a range of given term is called **classification of given term**.

The best known way of classification is **dichotomic classification**. Classification to elements that have or have not given feature. (Dichotomic means binary).

3 Examples of exercises for children

We will concentrate on the education itself:

1. Following exercise aims on creation of particular sets and childrens selection whether an object belongs to given set or not. From the mathematical point of view it concerns set and its element.

   Game: Children put on their necks pictures with various animals. Pictures prepares the teacher. It concerns animals living on land (lion, cat, dog etc.), aquatic animals (carp, pike etc.) and amphibian (frog). There are two colour circles on the ground. The teacher says: “Land animals go into the brown circle and aquatic animals go into the blue circle.” Among other things children practise the term circle and term inside circle and colours. Children check animals they represent and move into a circle of an appropriate colour. It arises a problem where to place the frog. Children will find out that circles can be crossed.

The above mentioned exercise can be modified e.g. with fruit and vegetable or vehicles - cars, trucks etc.

2. When teaching children propaedeutic of natural number we can do it e.g. in a following way: teach them to define groups of objects with the same number of elements based on assignment. The best way is pairing. Children identify in which group is more elements, less elements or the same number of elements. We can teach children regimented set of words: "one, two, three, four, five, six, seven, eight, nine, ten" by means of didactic rhymes and calculation e.g. "One little, two, little, three little indians . . . " etc. The aim is to make children to imagine a group of objects when they hear a certain word. Later on children should imagine name of object e.g. "House No. 3 on street No. 5".

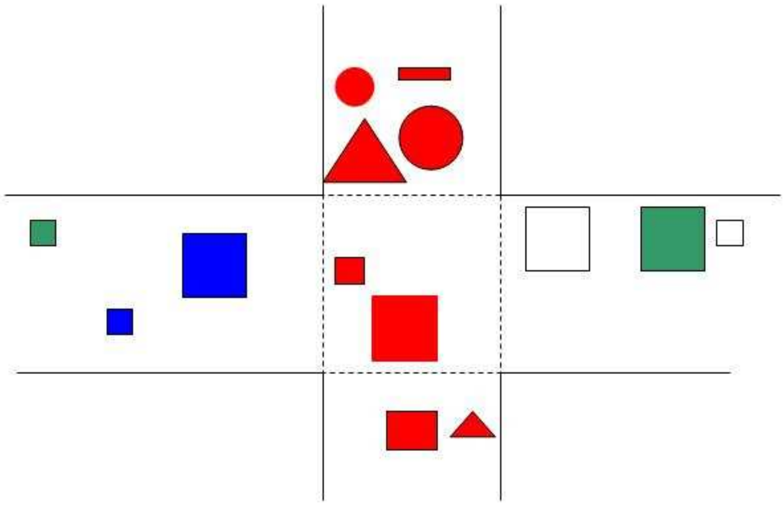
   Example of an exercise: On picture is mouse-mother and table with five plates. There are playing three mice-boys and three mice girls.

   Children are forming groups of objects based on given features and they decide which object belongs or does not belong to certain group. Then children form pairs – they allocate to one element from one group an element from a second group. (From the mathematical point of view it concerns simple scheme of sets). Teacher reads a text to children: While mouse-mother is cooking sweet porridge her little mice are playing together. Form a group af all little mice-children. Now mark a group of mice-girls and with different colour mark a group of mice-boys. Look at those two groups and say if there is more mice-girls or mice-boys. If you are not sure form pairs of boy and girl. Connect them with line. Count the number of mice-children. As the mice-mother finished the lunch she calls her children together. Give to each mouse-child one plate from the table (take a crayon and link up one mouse-child with one plate). Does each mouse-child get porridge or is it like in the children rhyme? Teacher reads to children the rhyme "Mouse cooked porridge".

3. We will practice geometrical figures and their allocation to groups.

   We will use real geometrical figures – magnetic triangles, squares, rectanglse and circles. Figures have four colours – mainly red, blue, green and white, and they have two sizes – big and small. In total it concerns 32 figures – small red triangles, big red triangles etc. (Those geometrical figures are known as Dienes's logic blocks).

   On magnetic table are drawn cross-roads. One street is called Square Str. and there live only squares, the other street is called Red Str. and there live only red geometrical figures. Children place to those streets magnetic geometrical figures. Teacher askes: "What is the name of the square where streets cross?" Children replies: "It is a Square of Red Square". The game continues in similar way (Triangle Street, Blue Street etc.).



4. Drawing game. By the means of drawing according to given instruction children get the idea of the meaning of terms: under, beside, above, right, left and they learn to place figures into the area of the picture.

   Teacher motivates children: “When I order a picture an artist usualy draws what I wish. Could you please draw a picture based on my wish? Draw a blue cloud into the middle of the picture, draw sun above the cloud, draw a house bellow the cloud, draw a tree left from the house and draw a bird flying to the sun. At what side of the house is empty space? Draw there a flower. If you like the picture take it home and hang it in your room.”

We presented some options of work with children.

References

[1] MELICHAR, J.; SVOBODA, J. *Rozvoj matematického myšlení I pro studium učitelství pro mateřské školy. (Development of Mathematical Thinking I for Education.)* Ústí nad Labem: Univerzita J. E. Purkyně, 2003. ISBN 80-7044-512-2.

[2] MELICHAR, J. *Rozvoj matematického myšlení. (Development of Mathematical Thinking.)* Ústí nad Labem: Univerzita J. E. Purkyně, 2010. Output of ESF project “We Can Do It Together – Programm of Pedagogical Intervention for Children from Social-cultural Handicaped Environment and Their Teachers”, reg. No. of the project: CZ.1.07/1.2.00/08.0105.

Development of Geometrical Imagination of Pupils

Jaroslav Perný,Jana Hanková,Tereza Nováková,
Tereza Votrubcová[1]

Technical University of Liberec,
Faculty of Science, Humanities and Education,
Department of Mathematics and Didactics of Mathematics

Abstract. Other possibilities of development of space imagination of pupils in primary school are discussed in this contribution. Many topics of mental manipulation with both plane and spatial objects, especially folding and decomposition of figures and solids, came into existence with help of our students of teaching primary school. Setting out the problems is also suitable and art integrated with help of these students.

Keywords. Geometrical imagination, problem solving, working sheets, spontaneus stereometry.

1 Introduction

The space imagination is one of the important human competencies useful for daily life and especially for some occupations. Research shows decrease of its level in our population. One of the possible reasons is also less attention given to this problem during school education. Several reasons can be found: preferring other themes as a result of decreased time donation for mathematics lessons, worse preparation of teachers in this field, but also an opinion, that the geometrical imagination is inherited and someone has it while someone has not. Although the last statement has some reasonable roots, each pupil's geometrical imagination can be strengthened and developed.

[1]The contribution was prepared within project SGS-FP TUL č. 5825/2010.

2 Methods and results

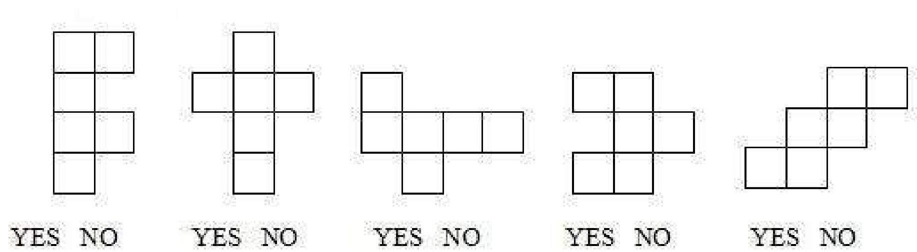
The possibilities of increasing the geometrical imagination of pupils include for example introduction of some themes by playful form in 1st grade of basic school, and as exercises and repeating in mathematics lessons in 2nd grade of basic school and on secondary school. These games and tasks are positively appreciated by pupils, but they require creativity and activity in teacher's work.

Consequently, it is very important to target these themes during university education of future teachers, offer themes and presentations of such problem tasks and playful activities to students, attract them to the problem and lead them towards their own creativity in the field. In the courses of didactically oriented subjects we tend to introduce to mathematics geometrical themes, mainly so called "spontaneous geometry", which students work upon within their semester thesis. We attempt to bring the students of primary school for this "different" geometry, so they can improve the abovementioned situation. Some of the student's works are very successful and show high level of creativity. It is nice to see, that number of students further develops their wok into master thesis. Some of the students continue these activities also in their teaching praxis after reaching the degree, or publish results of their creativity.

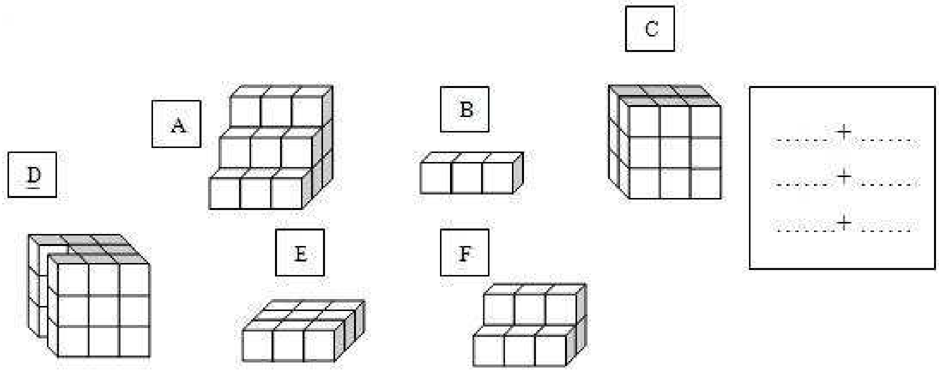
We would like to show partial results of our students' research in the field of development of geometrical imagination of pupils in schools, which they carry on during their praxis within the Student grant competition of TUL. In the attachments we present some examples of creative themes for geometrical imagination development which our students gained from us of from literature and further developed and which can be used in mathematical education on basic and secondary schools. The deal mainly with "spontaneous geometry" and they are divided into typological groups: nets of solids, composition and decomposition of solids (mainly cubical), projection of solids, hidden solids.

2.1 Example of test of geometrical imagination (Jana Hanková)

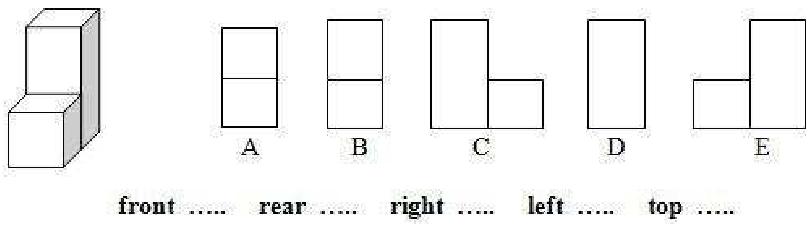
1. Out of which nets can be composed a cube? Circle YES or NO.



2. Which solids make together a cube?



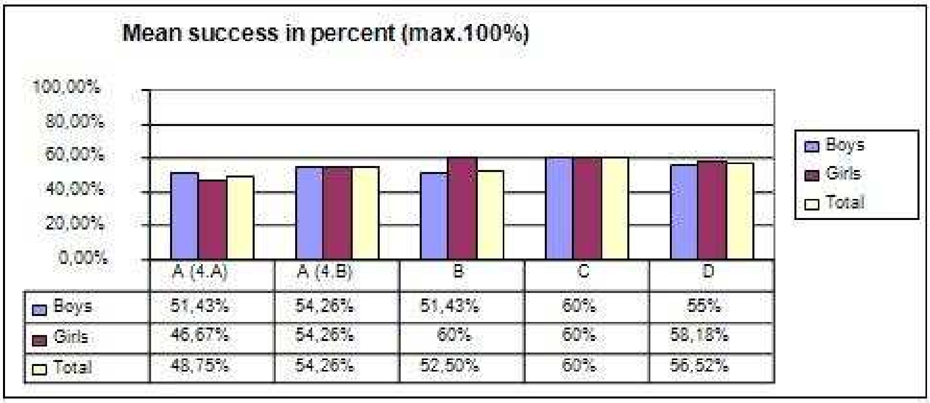
3. There are 5 views at the set of solids in the figures A-E. Link the correct statement to figures: **from front**, **from rear**, **from right**, **from left**, **from top**.



The test was performed at four schools. School A insmaller town, school B small-class, school C with Montessori education, school D complete village.

2.2 Results of test of geometrical imagination

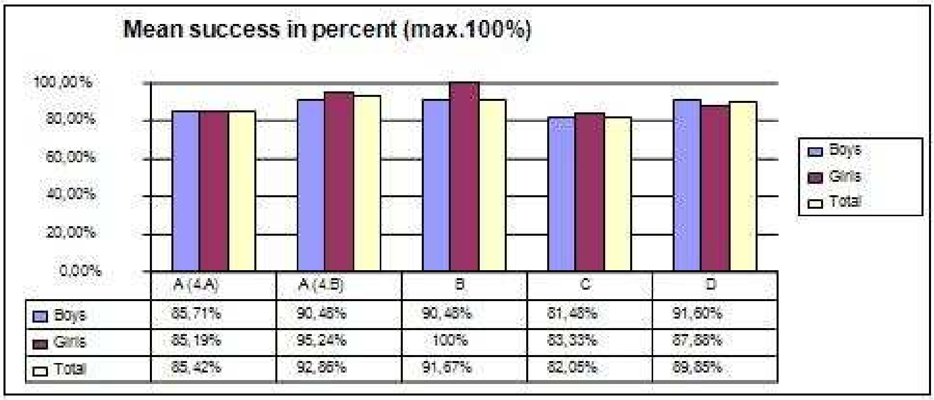
Task 1:



	A (4.A)	A (4.B)	B	C	D
Boys	51,43%	54,26%	51,43%	60%	55%
Girls	46,67%	54,26%	60%	60%	58,18%
Total	48,75%	54,26%	52,50%	60%	56,52%

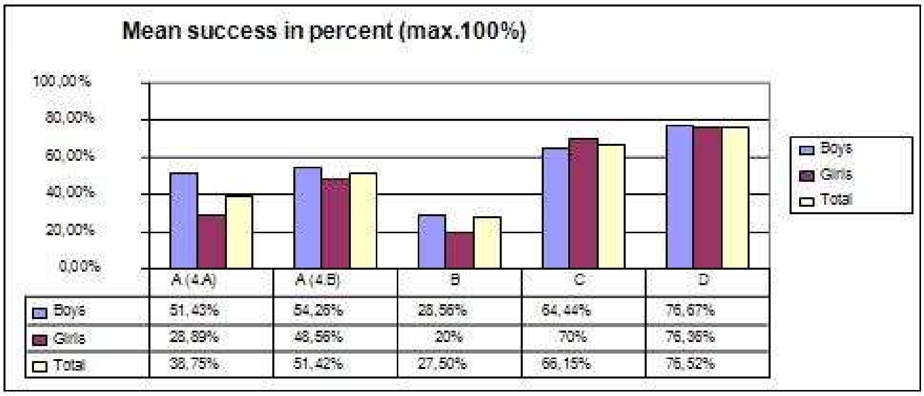
Relatively low success of pupils

Task 2:



Very high success of pupils

Task 3:



Quite variable success of pupils

3 Conclusion

We consider that presented examples of students' work show the potential of attracting students of primary school pedagogy (but also other grades) for geometry education. Together with sufficient offer of themes it is a possible way to improvement of our pupils' geometrical imagination. Creative teachers with this kind of preparation can successfully include these themes and develop them during education to motivate pupils and develop their imagination and other required competencies.

References

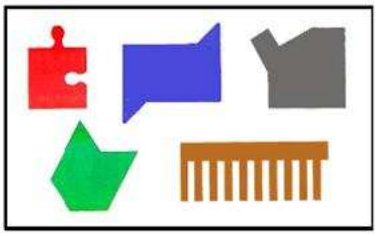
[1] PERNÝ, J. *Tvořivost k rozvoji prostorové představivosti.* Liberec: TU Liberec, 2004. ISBN 80-7083-802-7.

[2] PŘÍHONSKÁ, J. Tvořivost v matematice v přípravě učitele primární školy. *USTA ad Albim BOHEMICA*, 2009, vol. 9, no. 2., p. 29–47. ISSN 0146-5422.

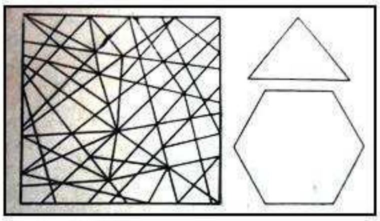
Attachments

Tereza Nováková - examples of worksheets for imagination development)

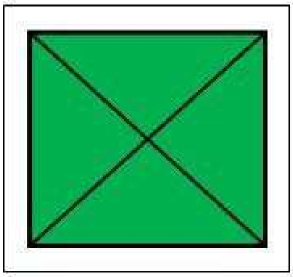
1. Divide figures by such single cut, to be able to create a square from the parts.



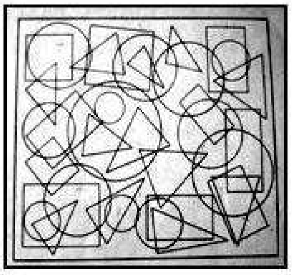
2. In the web of lines in the square, there are hidden, two figures from the right. Find them.



3. Cut square and create:
   from 2 pieces: triangle, square
   from 4 pieces: triangle, rectangle



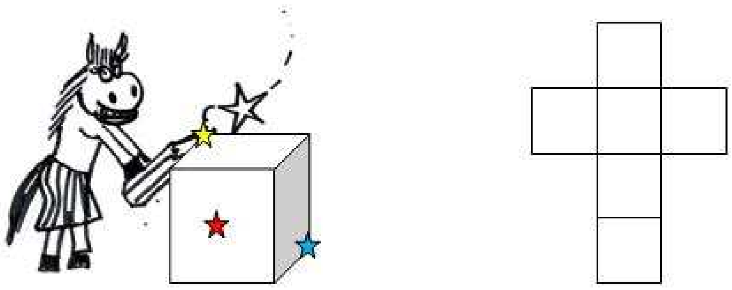
4. Find equal figures.



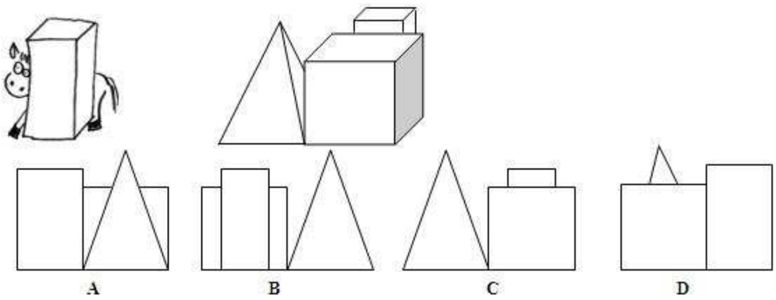
Jana Hanková - examples of worksheets for imagination development for 1st grade

Jana Hanková created for pupils helpful horse Emilka.

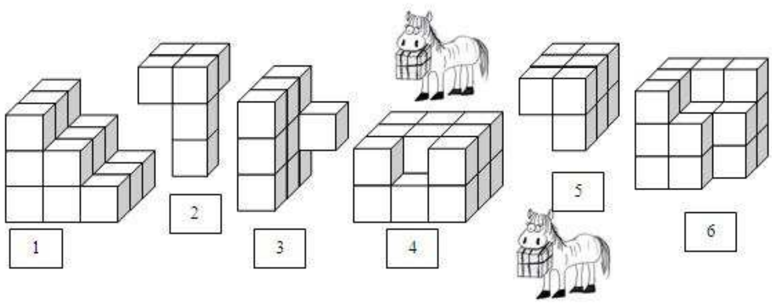
1. Emilka marked on cube three points. But her cube fell apart. Help her to add these points to the cube net to correspond with the first cube.



2. Emilka put together three solids and explored how they look like from various sides.



3. Emilka built one big cube out of small hay cubes. But her daughter Pepina during night broke the big cube apart. Help Emilka to find the correct couples that create a cube together.

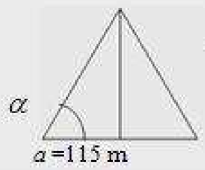


Tereza Votrubcová - examples of worksheets for imagination development for secondary school

Task “Pyramid”

Cheops’ pyramid is 138 m high. It has a square base with the length of the side approximately 230 m.
Calculate the inclination of Cheops’ pyramid walls.

Řešení:



v=138 m

$$\mathrm{tg}\ \alpha = \mathrm{tg}\,\frac{138}{115}$$

$$\alpha = 50°3\ \alpha$$

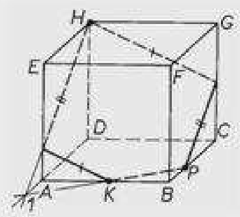
Task “Cut of the building”

Based on the cut of solid by plane, which students constructed themselves, was created a model of building.

Construct cut of cube $ABCDEFGH$ by plane HKP defined as:

$K = S_{AB}$; $P \in BC; |BP| : |PC| = 1 : 2.$

Solution:



Students’ city models

Expected Time for Tossing Heads and One Demographic Paradox

Adam Płocki

Instytut Matematyki, Uniwersytet Pedagogiczny Kraków,
Instytut Badań Pedagogicznych i Edukacyjnych SWPW Płock

Abstract. The aim of the paper is the determination of the value of expected random variable as a tool of a solid proof.

Keywords. Random variable, probability model, coin toss, waiting time.

1 Introduction

Let X be a random variable in denumerable (and discrete) probability space (Ω, p), and $\Omega_X = \{x_1, x_2, x_3, ...\}$ is a set of its values. For each $x_k \in \Omega_X$, set $\{\omega \in \Omega : X(\omega) = x_k\}$ is an event in the space (Ω, p). Let $P(X = x_k)$ denote its probability. If $x_k > 0$ for each $x_k \in \Omega_X$ and convergent number progression, then

$$\sum_{k=1}^{+\infty} x_k \cdot P(X = x_k),$$

is its sum s which is called *value of expected random variable* X is denoted with $E(X) = s$.

Let δ is a random event at the biggest possible set Ω_δ of all possible results, and p_δ – a function, which gives to every result the particular probability with which the given event δ can end. The pair $(\Omega_\delta, p_\delta)$ is called *probability model of event* δ.

2 Probability Models of Coin Toss and k-multiple Coin Toss

The coin toss is understood in this work as an random event δ_M with two the same likely results:

$$t \text{ -- } \textit{tails are tossed} \quad \text{and} \quad r \text{ -- } \textit{heads are tossed}$$

Pair (Ω_M, p_M), where $\Omega_M = \{t, h\}$ and $p_M(t) = p_M(h) = \frac{1}{2}$ is a classic probability space.

Result of a k-multiple coin toss is a k-term progression with elements from set $\{t, h\}$. Its j-th term is the result of the j-the toss, $j = 1, 2, 3, \ldots, k$. If Ω_M^k denotes a set of results of k-multiple toss coin, then $\Omega_M^k = \{t, h\}^k$. There are 2^k of all results of k-multiple coin toss and all are equally probable with probability equal to $\frac{1}{2^k}$.

3 Waiting for Heads and Its Probability Model

The repetition of coin toss until heads are tossed is a random event with a random number of phases. We call it *waiting for heads*. Let ω_k denote result:

$$\textit{heads is tossed for the first time in } k\textit{-th toss} \quad k = 1, 2, 3, \ldots.$$

Hence ω_k is the result of the k-the toss and therefore its probability is $\left(\frac{1}{2}\right)^k$. The probability model of waiting for heads (see [2], p. 16) is therefore pair (Ω_h, p_h), where

$$\Omega_r = \{\omega_1, \omega_2, \omega_3, \ldots\} \text{ and } p_r(\omega_k) = \left(\frac{1}{2}\right)^k \text{ for } k = 1, 2, 3, \ldots$$

Pair (Ω_h, p_h) is discrete probability space (see [4], p. 48).

4 Waiting Time for Heads and Its Value of Expectation

Let's say that consecutive coin tosses are made in consecutive time periods. Function T, which gives to the result of waiting for heads ω_k number k, is a random variable in the probability space (Ω_h, p_h). Its values form set $\Omega_T = \{1, 2, 3, \ldots\}$, and also

$$P(T = k) = \frac{1}{2^k} = \left(\frac{1}{2}\right)^k \text{ for } k = 1, 2, 3, \ldots$$

In [5] (p. 340) it is shown that $\displaystyle\sum_{k=1}^{+\infty} k \cdot \left(\frac{1}{2}\right)^k = 2$,
and then $E(T) = 2$. It is the mean time of waiting for heads in two coin tosses.

5 Sultan, Fighters, Mathematics and Stochastics

In [3] (p. 276) is given the following problem.

> *One ruler of a country, which was on permanent war, ordered that in the country (concerning all his subjects) should be more men than women. We could say today that it was "food for canons". He ordered that in*

each family can be only one girl. If there is, in any family, a girl born, the mother must not be pregnant again. It seems that the result of the order is obvious and in the country there are more men then women.

Let say that the probability that a boy or a girl is born is the same. We can simulate the born of the children by tossing a coin. The interpretation of one toss is that:

heads is tossed – a boy is born,
tails is tossed – a girl is born.

It is clear that the sex of a new born child is not dependent on the sex of the previous child. We can model the sex of k consecutive children by k-multiple coin toss.

Considering the order of the ruler regarding the number of children and their sex in all families in his country, is the repetition of coin tosses until heads are tossed (see [2], p. 254).

One waiting for tails simulates the history of one family. This means that the waiting for tails for n-times simulates a process of creating of children in n families.

Let us assume that n is a sufficiently long natural number. It looks like that the number of tails in n waiting for heads will be much bigger than the number of tossed heads. However, we can show that the assumption is not correct.

6 Expected Number of Tails in *n* Repetitions of Waiting for Heads

In n repetitions of waiting for heads, heads are tossed n-times. In n families, there are therefore n girls. The other children are boys. The number of all children in n families is random variable X_n. The number of coin tosses in j-th repetition of waiting δ_r is random variable T_j. It is the time of waiting for tails, and its value $E(T_j)$ is equal to 2 according to $j = 1, 2, 3, \dots, n$.

Random variable X_n is the sum $T_1 + T_2 + T_3 + \dots + T_n$, and so (see statement in [6], p. 224)

$$E(X_n) = E(T_1 + T_2 + T_3 + \dots + T_n) =$$
$$= E(T_1) + E(T_2) + E(T_3) + \dots + E(T_n) = n \cdot 2 = 2n.$$

The expected number of all children in n families is equal to $2n$, the number of girls equals n, and so the expected number of sons in n families is the expression $(2n - n)$, which makes n. Therefore, it can be assumed that:

– in n repetition of waiting for tails, the expected number of tossed heads is the same as the number of tossed tails;

– in the population of the families, it can be expected the same number of boys as the number of girls.

The ruler's intention does not ruin the balance in the population concerning the sex of new born children. It is a paradox as the intuition leads to a different conclusion.

7 Argumentation Based on Tables of Random Numbers

Let us assume tables of random decimal numbers (see [4], p. 205 or [7], p. 64 and pp. 526–528).

The probability that on the drawn number place will be an even number (resp. odd) is equal to $\frac{1}{2}$. A coin toss can be simulated using tables of random numbers. So it is possible to interpret even digits $(0, 2, 4, 6, 8)$ on the cards as “tails” (t), and odd digits $(1, 3, 5, 7, 9)$ as “heads” (h). The waiting for heads can be simulated by reading (from the drawn number place) the following digits until we come across an odd digit. The number of read digits is the number of coin tosses when waiting for heads.

Let us consider one (drawn) number of on a chosen page of the tables of random numbers in [7] on pp. 526–528. There is a progression of 1 750 digits (35 lines, 50 digits in each line). If we start reading the following digits from the very beginning, we will get the same numbers as waiting for heads. The expected number of odd digits in such a long progression of digits is equal to the number of tails. One page of the tables of random numbers represents particular families as much as. The number of odd numbers on this page is the number of girls, the number of even numbers on this page is the number of boys.

Therefore, it is justified to say that:

- in the described progression, there are more even digits than odd ones,
- the ruler’s command does not break the balance of sex of new born children in the families of his subjects.

References

[1] FELLER, W. *Wstęp do rachunku prawdopodobieństwa.* Warszawa: PWN, 1987.

[2] PŁOCKI, A. *Dydaktyka stochastyki.* Płock: Wydawnictwo Naukowe NOVUM, 2005.

[3] PŁOCKI, A.; TLUSTÝ, P. *Kombinatoryka wokół nas.* Płock: Wydawnictwo Naukowe NOVUM, 2010.

[4] PŁOCKI, A.; TLUSTÝ, P. *Pravděpodobnost a statistika pro začátečníky a mírně pokročilé.* Praha: Prometheus, 2007.

[5] PŁOCKI, A. *Pravdepodobnosť okolo nás – stochastika v úlochách a problémoch.* Ružomberok: Katolícka univerzita, 2007.

[6] PŁOCKI, A. *Prawdopodobieństwo wokół nas.* Wilkowice: Wydawnictwo Dla Szkoły, 2004.

[7] PŁOCKI, A. *Stochastyka dla nauczyciela.* Płock: Wydawnictwo Naukowe NOVUM, 2007.

Problem Solving in Interactive Environment

Jana Příhonská[1]

Technical University of Liberec,
Faculty of Science, Humanities and Education,
Department of Mathematics and Didactic of Mathematics

Abstract. In the report, a creative attitude during the teaching of mathematics regarding to a pupil's motivation and the use of modern didactic technologies, is discussed. We focus on the use of a presentation material and the SMART technologies during the process of analyzing some mathematic topics regard to a creation of stimulated interactive environment for the teaching of mathematics.

Keywords. Interactive environment, power point presentation, Smart Board, solving strategies.

1 Introduction

Students of teaching very often ask one particular question: "How to teach the pupils to think?" Asking that question, statement of particular problems, holding a discussion with pupils becomes a great problem. Students-future teachers have to learn how to work with pupils in problem solving and to develop their skills by using different solving strategies. The teachers motivate their pupils mainly by their own enthusiasm, for instance, by creating suitable tutorial materials which can be completed based on pupil's remarks and observations from the lessons. In the report, some examples of problems in interactive environment with the use of the interactive SMART Board are discussed. Some methodical notifications are prepared for the use of prepared material for a teacher as well.

[1]This research is supported by Projekt SGS - FP TUL - 5820

2 Integration of modern technologies in teaching mathematics

According to Žilková [3], the teaching of mathematics in a modern interactive environment is important in terms of curriculum (allowing real situations to be mathematizated, an interpretation of mathematic models in the real world, a global view to a selected part of subject matter by an application of integrated relationships in an information environment), pedagogic-psychological point of view (affecting the social climate in the class, a motivation, the use of creative elements) and a sociological point of view (monitoring of a final effect of a pedagogic-educational process with the use of a modern information technology (IT) in the real world.

If we want to educate young people how to be prepared to entry the reality, we have to teach them interestingly with the use of different motivating activities and our own creative ideas, materials and tools, [2]. It is necessary to start from a file of activities based on the pupils' own cognitive and creative activity. It is obvious that this activity is organized and directed by the teacher. The active participation allows creative work to take place – it evolves the ability of a logical and analytic thinking plus certain aesthetic standards are developed as well. A certain aesthetic moment can play a big role in a mathematical thinking. A mathematician must be able to create their own imagination of a situation that they are concerned with and to work with it as it was the reality. In other words they must be able to image a world that is represented by some formulas in the end. Most of the time, it is true that the final solutions are simple, elegant and beautiful. Of course those beautiful and simple solutions are not available for everyone – just like music, it demands some background.

The way mathematics is taught can reflect as pupils' aversion to mathematics. Mathematics, and especially the teachers of mathematics, did a pretty good job with discouraging many students who could eventually be concerned with mathematics by using too difficult and mechanical ways how to solve problems. It is impossible to want students to solve ridiculous problems or to memorize facts. A teacher ought to invoke an interest in mathematics, try to show what a mathematic thinking is and how it can be useful. Subject matter can be explained much better with the use of motivating examples than with the use of boring phrases such as "Let an x be ..." Mathematics is losing its potential supporters and users. On the other side an interpretation like that must not turn into "pointless crap".

2.1 Motivation

In this context, the use of computer technology is a debatable question. We must point out that, in mathematics, computers are currently the main source of communication and getting information. Though, they are not that important as a research tool for the development of mathematics itself. With the exception of some areas as combinatory and the number theory, so far, computers do not enable obtaining new and obligatory results. Mathematics mostly operates with the infinity and abstraction, and that is what a computer is not capable of doing. But computers opened a new and a wide field of applications. Thanks to them a new scientific area, mathematical informatics, was developed.

But for sure, the biggest profitability of computers is that they allow their users to get rid of the necessity of practising some routine mathematical operations. Though, it must be pointed out that everyone who uses a computer for that reason must be able to understand all the operations behind the results and that is why mathematics must not disappear from the curriculum.

The role of a teacher is, by a properly choice of motivation, an interesting explanation (with an illustration, for example), a way of running the lessons, etc., to take care of the motivating elements that negatively affect pupils' performance, to eliminate a potential source of pupils' negative motivation, and to prevent frustration which could lead to the elimination of their effort and studying goals. By that, we mean boredom and fear. Among the featured motivational factors, we can find evaluating and a classification, motivation as the purpose of the class, a quality of the learning process, and matter of interest of the curriculum, a personality of the teacher, a social feedback in the classroom (and out of it as well), emotional relationships of a pupil with the environment, the teacher, the classmates, etc. We can also add other elements that differ depending on what the motivation is intended on or on what it is supposed to be good for:

- interesting mathematical warm-ups
- introduction of new subject matter
- getting the attention during the learning process – "waking the pupils up"
- awaking the curiosity – "a shocking result"
- opening of the problem to think about
- comparison of different solving strategies of a certain problem – opening a discussion
- connecting different areas of mathematics – Fig. 1 (left)
- application of mathematics in the real-world – Fig. 1 (right)
- application of mathematics in other subjects – Fig. 1 (left)
- relaxing elements – interesting elaboration of a topic (music background)
- interesting didactic games – their elaboration
- the use of the internet – searching for new information

We use the motivating factors mentioned above with students of the teaching of mathematics or with students of elementary-school teaching during the elaboration of mathematical problems or certain mathematical units with the use of Power Point presentation. These presentations are then elaborated in the interactive environment of the SMART Board.

2.2 Teaching mathematics in interactive environment

The use of information technology allows us to set up the investigated phenomenon in different situations and it has an influence on different senses of the recipients.

Figure 1. Motivation.

In this context, we are talking about an interactive board. The interactive board is a piece of equipment that makes it possible to transform a "normal" white-plastic board to a touch screen via which it is possible to control the compute and its applications. One of the other characteristics is the possibility of recording and elaboration user's notes, taken by an electronic pen, and transform then into a digital form that can be used at a different time. Both variations support the realization of a more effective education with the ability of using new educational methods and procedures. Recipients can participate in making the scenario of the lessons, come up with hypothesis and then they can also correct and develop it.

New options for cooperation, complementation, interaction, and a realization of one's own ideas are being opened in this field. With the help of the interactive board, we can use different multimedia products, animation techniques, browse free on the internet, gain information, and realize all the activities that we usually realize while using a computer. In the context of spreading interactive products, the method of "sharing applications or files" [3], in real time, is classified within one room, building, or eventually within a much bigger distance. Also, the interactive environment makes it possible to apply virtual manipulative methods in the teaching of mathematics (the models of a subject manipulation are virtual in this context).

An interactivity is a general requirement for teaching, especially for the teaching of mathematics. The development of IKT technologies makes it possible to effectively use the interactive equipment and the applications on different educational levels – both visual and demonstrative examples in the interactive environment, a pupil's experience, and manipulation with interactive products, their own proposal, or participation during the elaboration of mathematical interactive outputs. In this respect, potential concerned persons should see a monograph of K. Žilková [3] - in the book, she describes many methods and devices for the use of interactive tools in the teaching of mathematics, including practical instruments and links.

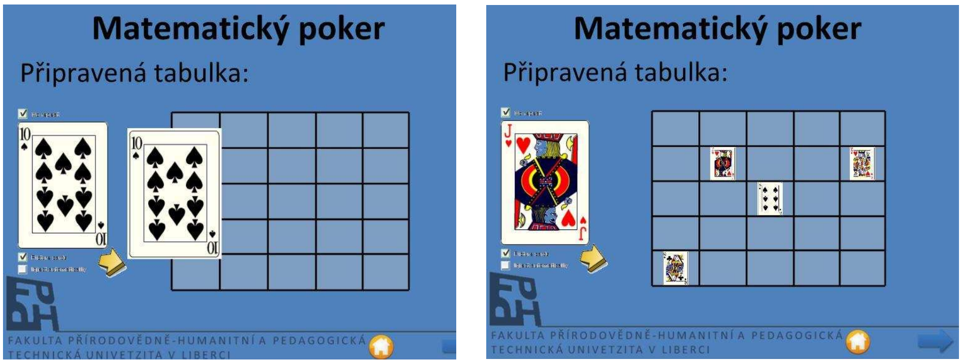


Figure 2. Generated cards (left) and setting the cards (right).

2.3 Problem Solving in SMART Board Environment

During preparing future teachers at FP TU in Liberec, we focus, among others, on the development of solving strategies of given problems, motivation and the presentation. The goal is the development of individual creative abilities of the future teachers which takes shape as the ability of solving the problems initially, unexpectedly, sagaciously, and surprisingly without using already existing processes but with the use of their own new ideas. By that the students, future teachers, bring certain "newness" to the teaching. We consider that as the essential qualitative character of creativity. Besides the "newness", we also focus on different motivational factors affecting the pupil's positive attitude towards mathematics.

Let's illustrate the offered options on one of mathematical games. There is a setting of a more-or-less known game, math poker. The goal of this game is to, by a random choice of numbers, fill the prepared table and to, after assigning points based on the given rules, gain a maximum amount of points. Often, instead of dictating the winning numbers, a card pack is used.

The SMART Board interactive environment can take advantage of new opportunities. The cards are randomly generated (Fig. 2, left), a certain card/number can be filled directly in the table (Fig. 2, right), or you can just simply write down the value of the generated cards. One student works directly with the board, others work independently. For the students, this performance of a game becomes more interesting and encourages them to actively participate in the realization of the game. Other ideas, how to vary the game, may appear. Plenty of other games can be elaborated in the environment of the interactive board. You can include clues that a student can use, when needed. You can also graduate the difficulty of the problems depending on the age and abilities of the student.

3 Conclusion

In collaboration with students of teaching at FP TU in Liberec, we make Power Point presentations for a whole set of problems, games, and ideas for working with pupils on the first, respectively the second, level of elementary school. Many of those problems are further modified to be used in the interactive environment of the SMART Board and will be offered to schools for further use in educating.

References

[1] PŘÍHONSKÁ, J. *Prezentační software v přípravě budoucích učitelů.* In Žilková, K. *Potenciál prostredia IKT v školskej matematike.* (CD), Univerzita Komenského v Bratislave, 2009, p. 44-53. ISBN 978-80-223-2754-1.

[2] SCHOLTZOVÁ, I. Jedna cesta od odborných k didaktickým kompetenciám učitel'a – elementaristu. In *ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS, FACULTAS PAEDAGOGICA, MATHEMATICA V. Matematika 2. Matematika jako prostředí pro rozvoj osobnosti žáka primární školy. Sborník příspěvků z konference s mezinárodní účastí.* Olomouc: UP, 2006. p. 224 - 228. ISBN 80-244-1311-6.

[3] ŽILKOVÁ, K. *Školská matematika v prostredí IKT (Informačné a komunikačné technológie).* Univerzita Komenského Bratislava, 2009. ISBN 978-80-223-2555-4.

Several Remarks on the Field of Fractions of Integer Rings

Grażyna Rygał[1], Arkadiusz Bryll[2]

[1] Jan Długosz University in Częstochowa,
Pedagogical Department

[2] Technical University of Częstochowa,
Management Department

Abstract. In this paper we formulate criteria concerning existence of common field of fractions for two integer rings and determine the properties of field of fractions for an arbitrary field.

Keywords. Field of fractions, integer ring, ring isomorphism.

1 The field of fractions of an integer ring

A commutative-associative ring without zero divisors is called an integer ring. Hence, we do not assume that in the integer ring there is an identity element for multiplication.

A field of fractions of integer ring $(A, +, \cdot)$ is built as follows:
In a set $A^* = A \times (A \backslash \{0\})$ we define a relation R and two operations $\oplus$ and $\odot$ and assume that

$$(a, b)R(c, d) = a \cdot d + b \cdot c, \tag{1}$$

$$(a, b) \oplus (c, d) = (a \cdot d + b \cdot c, b \cdot d), \tag{2}$$

$$(a, b) \odot (c, d) = (a \cdot c, b \cdot d). \tag{3}$$

The relation R is a congruence, hence we can construct abstraction classes and a product algebra $(A^*/R, \boxplus, \boxdot)$, where the operations $\boxplus$ and $\boxdot$ are defined as follows:

$$\langle a, b\rangle_R \boxplus \langle c, d\rangle_R = \langle (a, b) \oplus (c, d)\rangle_R,$$
$$\langle a, b\rangle_R \boxdot \langle c, d\rangle_R = \langle (a, b) \odot (c, d)\rangle_R.$$

An element $\langle 0, b\rangle_R$ of a product algebra is an identity one for the operation $\boxdot$. An abstraction class $\langle a, b\rangle_R$ is called a fraction and usually is denoted as $\frac{a}{b}$. An element opposite to a fraction $\frac{a}{b}$ is a fraction $\frac{-a}{b}$, whereas an element inverse to a fraction $\frac{a}{b}$ is a fraction $\frac{b}{a}$ if $a \neq 0$.

2 A problem of existense of a common field of fractions for two integer rings

It was shown in [2] that:

Theorem 1. *Integer rings $(A, +, \cdot)$ and $(A, +, \circ)$ with the same additive group $(A, +)$ have a common field of fractions if and only if the following condition is fulfilled:*

$$\exists_{\alpha,\beta\in A,\alpha\neq 0,\beta\neq 0}\left\{\forall_{a,b\in A}[\alpha\cdot(a\circ b) = \beta\cdot a\cdot b] \vee \forall_{a,b\in A}[\alpha\circ a\circ b = \beta\circ(a\cdot b)]\right\}. \tag{I}$$

If $\beta = \lambda\cdot\alpha$ or $\alpha = \lambda\cdot\beta$ (where $\lambda \in A$), then the condition (I) becomes

$$\exists_{\lambda\in A,\lambda\neq 0}\left[\forall_{a,b\in A}(a\circ b = \lambda\cdot a\cdot b) \vee \forall_{a,b\in A}(a\cdot b = \lambda\circ a\circ b)\right]. \tag{II}$$

Theorem 2. *If an operation "$\circ$" in an integer ring $(A, +, \cdot)$ is defined by a formula*

$$a\circ b = \lambda\cdot a\cdot b \quad (a, b \in A), \tag{4}$$

where λ is a fixed element of a set A differing from zero, then

a) a structure $(A, +, \circ)$ is also an integer ring,

b) integer rings $(A, +, \cdot)$ and $(A, +, \circ)$ have a common field of fractions.

Equation (4) is valid, for example, in a ring $(C^1, +, \cdot)$, where C is a set of functions having continuous first derivative defined in an interval $[0, +\infty)$ with values in a set of complex numbers, whereas the operations "+" and "$\cdot$" are defined as follows:

$$h = f + g \Leftrightarrow \forall_{t\geq 0}[h(t) = f(t) + g(t)],$$
$$h = f\cdot g \Leftrightarrow \forall_{t\geq 0}[h(t) = \frac{d}{dt}\int_0^t f(t-x)\cdot g(x)dx] \quad (f, g, h \in C^1).$$

If in a set C^1 we define a convolution operation "$*$"

$$h = f * q \Leftrightarrow \forall_{t\geq 0}\left[h(t) = \int_0^t f(t-x)\cdot g(x)dx\right],$$

then the condition (4) is fulfilled, and we have

$$f * g = \lambda \cdot f \cdot g, \tag{5}$$

where $\lambda(t) = t$ for an arbitrary $t \in [0, +\infty)$.

The rings $(C^1, +, \cdot)$ and $(C^1, +, *)$ are not isomorphic as there is no identity element for the convolution operation "$*$".

The common field of fractions for those rings is known as the field of Mikusiński operators. Mikusiński constructed a field of operators for the purpose of presenting a method for solving linear differential equations. This metod distincts from those based on integral transforms, for example, based on the Laplace transform. Extensive information about the field of Mikusiński operators can be found in [1–6].

The condition (I) may be also fulfilled for integer rings with the same additive group in which any of the "multiplication" operations is not defined by the rest. For example, if in the ring of integers $(Z, +, \cdot)$ with the standard addition and multiplication operations we define the operations $\triangle$ and $\bigtriangledown$ as follows:

$$a \triangle b = \lambda_1 \cdot a \cdot b, \tag{6}$$
$$a \bigtriangledown b = \lambda_2 \cdot a \cdot b, \tag{7}$$

where $\lambda_1, \lambda_2 \in Z$ and none of this numbers is the multiple of another, then the following relations are fulfilled in integer rings $(Z, +, \triangle)$, $(Z, +, \bigtriangledown)$

$$\lambda_2 \bigtriangledown (a \triangle b) = \lambda_1 \bigtriangledown (a \bigtriangledown b), \tag{8}$$
$$\lambda_1 \triangle (a \bigtriangledown b) = \lambda_2 \triangle (a \triangle b). \tag{9}$$

Hence, the condition (I) is satisfied.

It should be noted that the ring of integers $(Z, +, \cdot)$ has the following property:

Theorem 3. *If in the ring of integers* $(Z, +, \cdot)$ *with the standard addition and multiplication operations the operation "$\circ$" is definable by the operations in this ring, then a structure* $(Z, +, \circ)$ *is a ring and the condition*

$$\exists_{\lambda \in Z, \lambda \neq 0} \forall_{a,b \in Z} [a \circ b = \lambda \cdot a \cdot b] \tag{III}$$

is satisfied.

Proof of this theorem is given in [2].

3 Existence of a common field of fractions for isomorphic ring over the same universe

It is known that there are isomorphic integer rings over the same universe for which the fields of fractions are not identic but only isomorphic. As an example we can consider the ring of integers $(Z, +, \cdot)$ with the standard addition and multiplication operations and the integer ring $(Z, \vee, \wedge)$ with the operations defined as follows:

$$a \vee b = a + b + \beta, \tag{10}$$

$$a \wedge b = a \cdot b + \beta(a + b) + \beta \cdot (\beta - 1), \tag{11}$$

where $a, b, \beta \in Z, \quad \beta = \text{const}$.

For $\beta = 1$ we obtain the well-known example of a ring with operations:

$$a \vee b = a + b + 1, \tag{12}$$

$$a \wedge b = a + b + a \cdot b. \tag{13}$$

An isomorphism is defined by the function $\Phi : Z \to Z$ of the form

$$\Phi(x) = x + 1.$$

Consider two integer rings $(A, \vee, \wedge)$ and $(A, +, \cdot)$ over the same universe.

It is easy to show that

Theorem 4. *Functions* $\Phi_i : A \to A \quad (i = 1, 2, 3, 4)$ *with values*

$$\Phi_1(x) = x + \beta \quad (\beta = const), \tag{14}$$

$$\Phi_2(x) = \alpha \cdot x \quad (\alpha \neq 0), \tag{15}$$

$$\Phi_3(x) = \alpha \cdot x + \beta, \tag{16}$$

$$\Phi_4(x) = \alpha \cdot (x + \beta) \tag{17}$$

transform a ring $(A, \vee, \wedge)$ *into a ring* $(A, +, \cdot)$ *isomorphically if and only if the operations in these rings satisfy the conditions:*

$$\begin{aligned} a \vee b &= a + b + \beta, \\ a \wedge b &= a \cdot b + \beta \cdot (a + b) + \beta^2 - \beta, \end{aligned} \tag{14'}$$

$$\begin{aligned} a \vee b &= a + b, \\ a \wedge b &= \alpha \cdot (a \cdot b), \end{aligned} \tag{15'}$$

$$\begin{aligned} a \vee b &= a + b + \alpha^{-1} \cdot \beta, \\ a \wedge b &= \alpha \cdot (a \cdot b) + \beta \cdot (a + b) + \alpha^{-1} \cdot (\beta^2 - \beta), \end{aligned} \tag{16'}$$

$$\begin{aligned} a \vee b &= a + b + \alpha \cdot \beta, \\ a \wedge b &= \alpha \cdot (a \cdot b) + \alpha \cdot \beta \cdot (a + b) + \alpha \cdot \beta^2 - \beta. \end{aligned} \tag{17'}$$

It follows from Theorem 4 that if the operations in integer rings $(A, \vee, \wedge)$ and $(A, +, \cdot)$ satisfy the conditions (15′), then a function $\Phi_2 : A \to A$, $\Phi_2(x) = \alpha \cdot x$ $(\alpha \neq 0)$ is an isomorphism. According to Theorem 2, the above-mentioned rings have the common field of fractions. In the case of isomorphism $\Phi_1(x) = x + \beta$ $(\beta \neq 0)$ the considered rings do not have the common field of fractions.

We shall prove that

Theorem 5. *If a function* $\Phi_3(x) = \alpha \cdot x + \beta$ *or a function* $\Phi_4(x) = \alpha(x + \beta)$*, where* $\alpha, \beta \in A (\alpha \neq 0)$ *is an isomorphism of an integer ring* $(A, \vee, \wedge)$ *and an integer ring* $(A, +, \cdot)$ *and* $|A| \geq 3$*, then these rings have the same field of fractions if and only if* $\beta = 0$*.*

Proof:

"$\Leftarrow$" It follows from Theorem 4 that for isomorphisms Φ_3 and Φ_4 with $\beta = 0$ the operations in the considered rings satisfy the following conditions: $a \vee b = a + b$, $a \wedge b = \alpha \cdot (a \cdot b)$ (see (16′) and (17′)). According to Theorem 2, the integer rings $(A, \vee, \wedge)$, $(A, +, \cdot)$ have the common field of fractions.

"$\Rightarrow$" Now we prove that for $\beta \neq 0$ the fields of fractions of the given rings are different. Suppose that $\beta \neq 0$ and consider a set $A^* = A \times (A \backslash \{0\})$ with the relation S defined as follows:

$$(a, b) S (c, d) \Longleftrightarrow a \wedge d = b \wedge c. \tag{18}$$

We will show that the relation S and the relation R defined by (1) are different. For this purpose take two elements $a, b \in A$ such that $a \neq 0$, $b \neq 0$, $a \neq b$. Such elements exist based on the condition $|A| \geq 3$. In this case we have:

$$(0, a) R (0, b), \quad \text{since} \quad 0 \cdot b = a \cdot 0.$$

If $a \neq b$, then $\Phi_i(a) \neq \Phi_i(b)$ $(i = 3, 4)$ and $\Phi_i(0) \cdot \Phi_i(a) \neq \Phi_i(0) \cdot \Phi_i(b)$, as $\Phi_3(0) = \beta \neq 0$ and $\Phi_4(0) = \alpha \cdot \beta \neq 0$. From this fact, it follows that $\Phi_i(0 \wedge a) \neq \Phi_i(0 \wedge b)$ $(i = 3, 4)$ and as a consequence $0 \wedge a \neq 0 \wedge b$, i.e. $0 \wedge b \neq a \wedge 0$.

According to (18), we have $\sim [(0, a) S (0, b)]$. Since $R \neq S$, then we get $A^*/R \neq A^*/S$. Hence, for the rings $(A, \vee, \wedge)$ and $(A, +, \cdot)$ we obtain two different fields of fractions. For example, for the integer fields $(Z, \vee, \wedge)$ and $(Z, +, \cdot)$ with operations satisfying the conditions (16′) or (17′) with $\beta \neq 0$ the fields of fractions are different.

4 A field of fractions for the ring of even integers

Consider the structure $(Z_0, +, \cdot)$, where Z_0 is a set of all even integers with the standard addition and multiplication operations.

The structure described above is a ring without identity. However, it is possible to construct a field of fractions for this ring.

We introduce the realtions R_1 and R_2 in the sets $Z^* = Z \times (Z \backslash \{0\})$ and $(Z_0)^* = Z_0 \times (Z_0 \backslash \{0\})$, respectively, by definitions:

$$(a, b) R_1 (c, d) = a \cdot d = b \cdot c, \quad (a, b), (c, d) \in Z^*,$$
$$(a, b) R_2 (c, d) = a \cdot d = b \cdot c, \quad (a, b), (c, d) \in (Z_0)^*$$

It is evident that $R_2 = R_1 / Z_0$.

A field of fractions $(Z^*/R_1, \boxplus, \boxdot)$ is the field of rational numbers, whereas a field of fractions $((Z_0)^*/R_2, \boxplus, \boxdot)$ is isomorphic but not identic to the field of rational numbers. In fact, a function $\Phi : Z^*/R_1 \to (Z_0)^*/R_2$ with values $\Phi\left(\langle a,b\rangle_{R_1}\right) = \langle 2a, 2b\rangle_{R_2}$ is a bijective mapping satisfying the conditions of being isomorphism.

Hence, we have:

$$\begin{aligned}
\Phi\left(\langle a,b\rangle_{R_1} \boxplus \langle c,d\rangle_{R_1}\right) &= \Phi\left(\langle ad+bc, bd\rangle_{R_1}\right) = \langle 2(ad+bc), 2bd\rangle_{R_2} = \\
&= \langle 4(ad+bc), 4bd\rangle_{R_2} = \langle 2a, 2b\rangle_{R_2} \boxplus \langle 2c, 2d\rangle_{R_2} = \\
&= \Phi\left(\langle a,b\rangle_{R_1}\right) \boxplus \Phi\left(\langle c,d\rangle_{R_1}\right); \\
\Phi\left(\langle a,b\rangle_{R_1} \boxdot \langle c,d\rangle_{R_1}\right) &= \Phi\left(\langle ac, bd\rangle_{R_1}\right) = \langle 2ac, 2bd\rangle_{R_2} = \\
&= \langle 4ac, 4bd\rangle_{R_2} = \langle 2a, 2b\rangle_{R_2} \boxdot \langle 2c, 2d\rangle_{R_2} = \\
&= \Phi\left(\langle a,b\rangle_{R_1}\right) \boxdot \Phi\left(\langle c,d\rangle_{R_1}\right).
\end{aligned}$$

5 A field of fractions of an arbitrary field

An arbitrary field is an integer ring, therefore it is possible to construct a field of fractions for this field.
Note that for an arbitrary field $(K, +, \cdot)$ a quotient set K^*/R has the form

$$K^*/R = \{\langle a, 1\rangle; \quad a \in K\}.$$

Lemma 1. *A set K^*/R is constructed from abstraction classes $\langle a, 1\rangle_R$, where $a \in K$, and 1 is an identity of a field K.*

For example, for the residue field modulo 5, i.e. for a field $(Z_5, \overset{+}{(5)}, \overset{\cdot}{(5)})$ a set Z_5^*/R is constructed from the following abstraction classes:

$$\begin{aligned}
\langle 0,1\rangle_R &= \{(0,1),(0,2),(0,3),(0,4)\}, \\
\langle 1,1\rangle_R &= \{(1,1),(2,2),(3,3),(4,4)\}, \\
\langle 2,1\rangle_R &= \{(2,1),(4,2),(1,3),(3,4)\}, \\
\langle 3,1\rangle_R &= \{(3,1),(1,2),(4,3),(2,4)\}, \\
\langle 4,1\rangle_R &= \{(4,1),(3,2),(2,3),(1,4)\}.
\end{aligned}$$

Lemma 1 follows from the property

$$\langle a,b\rangle_R = \langle a\cdot b^{-1}, 1\rangle_R \quad \text{for} \quad (a,b) \in K^*.$$

Hence, in the field of fractions we have for arbitrary $a, b \in K$:

$$\begin{aligned}
\langle a,1\rangle_R \boxplus \langle b,1\rangle_R &= \langle a+b, 1\rangle_R, \\
\langle a,1\rangle_R \boxdot \langle b,1\rangle_R &= \langle a\cdot b, 1\rangle_R.
\end{aligned}$$

The following theorem is valid:

Theorem 6. *The field of fractions of an arbitrary field is isomorphic to this field.*

The stated theorem follows from the fact that a function $\varphi : K \to K^*/R$ with the values $\varphi(a) = \langle a, 1\rangle_R$ is an isomorphism since it is a bijective mapping and satisfies the conditions:

$$\varphi(a+b) = \langle a+b, 1\rangle_R = \langle a, 1\rangle_R \boxplus \langle b, 1\rangle_R = \varphi(a) \boxplus \varphi(b),$$

$$\varphi(a \cdot b) = \langle a \cdot b, 1\rangle_R = \langle a, 1\rangle_R \boxdot \langle b, 1\rangle_R = \varphi(a) \boxdot \varphi(b),$$

$$\varphi(0) = \langle 0, 1\rangle_R, \quad \varphi(1) = \langle 1, 1\rangle_R.$$

Therefore, it is obvious that according to Theorem 6 the construction of a fraction field is expedient only when the integer rings are not fields.

Such a construction is considered, for example, when a field of rational numbers is built based on a ring of integers and when a field of rational functions is built based on an integer polynomial ring.

References

[1] BERG L. *Einfürung in die Operatorenrechnung.* Berlin, 1962.

[2] BRYLL G.; BLIDIA M. O ciałach ułamków pierścieni całkowitych. *Zeszyty Naukowe WSP w Opolu, Matematyka*, 1990, 27, 79–86.

[3] BRYLL G.; ONISZCZYK R. Sposoby konstrukcji ciała operatorów Mikusińskiego. *Zeszyty naukowe WSI w Opolu, Problematyka różna*, 1972, 2, 3–15.

[4] MIKUSIŃSKI J. Sur les fondements du calcul opératoire. *Studia Matematica*, 1950, 11, 41–70.

[5] MIKUSIŃSKI J. *Rachunek operatorów.* Wyd. 2, Warszawa: PWN, 1957.

[6] RAJEWSKI M. Berechnung der Einschaltvorgänge in linearen Schaltungen mittels “abgeschnitterer” Funktionen. *Wiss Z. Hochsch. f. Elektrotechnik Ilmenau*, 1958, 4, 143–165.

[7] SIERPIŃSKI W. *Zasady algebry wyszej.* – Monografie matematyczne t. XI, Warszawa–Wrocaw, 1946.

GEOMETRY IN ELEMENTARY TEACHER TRAINING

Iveta Scholtzová, Marek Mokriš[1]

University of Prešov,
Faculty of Education, Department of Mathematic Education

Abstract. The Programme of International Student Assessment PISA, by assessing young people in their competences in reading, mathematics and science, determines how prepared for life they are. There are four overarching concepts or ideas under mathematical literacy framework one of which is Space and Shape. This area has ties to spatial and geometric phenomena and relations. In the mathematical curriculum such content area is taught as geometry. Geometry is an inseparable part of mathematical training of prospective elementary teachers. It is included in several courses of both undergraduate and graduate form of study. The research conducted within scientific project VEGA MŠ a SAV 1/0192/08 *Analysis of the mathematical training of students in Pre-school and Elementary Education programme from the aspect of developing mathematical literacy* includes a diagnostics of the standard of student - trainees mathematical literacy. One of the five test assignments is taken from the Shape and Space domain.
Quantitative and qualitative analysis of students' solutions of mathematical (geometry) tasks provides a set of data for adequate teaching strategy in terms of attaining the goals of mathematical training of elementary teacher trainees.

Keywords. Geometry, elementary teacher training, mathematical literacy.

1 Geometry in primary education

Since 2008/2009 academic year there has been a curricular transformation in Slovakia which affected primary education. According to the *State Programme of Education (national curriculum) Mathematic, annex 1, 2. adjusted version for 1st to 4th year of basic school* (ŠVP-M-ISCED1, 2009, p. 3) **Geometry and Measurement** is listed as one of the five thematic areas (p. 3). Within the given area pupils are

[1]This research is supported by scientific project VEGA MŠ a SAV 1/0192/08 *Analysis of the mathematical training of students in Pre-school and Elementary Education programme from the aspect of developing mathematical literacy.*

expected to create three-dimensional geometric figures according to given rules, to acquaint themselves with the drawing of the most common planar figures, to know basic properties of geometric figures, to compare, estimate and measure the length, to acquaint with the measures of the length and to solve adequate metric tasks from practice.
The pupil should acquire the following competences (ŠVP-M-ISCED1, 2009, p.11):

- *discerns, names, models and describes relevant basic three-dimensional geometric figures, is able to find their representation in practical reality,*
- *discerns, is able to describe, name and draw basic planar geometric figures,*
- *discerns and models simple symmetrical figures in plane,*
- *is familiar with length measurement tools and their units, is able to use them independently in practical measurement tasks.*

The syllabus of the **Geometry and Measurement** thematic area for each year (ŠVP-M-ISCED1, 2009, pp. 5 - 9):

1st year (recommended number of units 12)
Drawing lines. Drawing straight lines. Geometric figures and formations - drawing. Manipulations with selected three-dimensional and planar geometric figures.

2nd year (recommended number of units 20)
Point, line, half line, line segment. Drawing lines and segments. Marking segments on line, half line and geometric figure. Units of length - cm, dm, m. Measurement of segments length. Comparing segments according to their length. Constructing solids from blocks according to pattern or picture. Constructing simple solids.

3rd year (recommended number of units 20)
Measurement of segments length in mm a cm. Measurement of larger distances: estimate (by footsteps) with precision in metres. Estimate of length: smaller in cm (mm); larger in metres. Drawing - essential rules of drawing. Drawing lines and segments. Marking segments on line or given geometric figure. Drawing planar formations in grid. Enlarging and decreasing size of planar formations in grid. Constructing solids from blocks according to plan (picture). Drawing plane of construction from blocks.

4nd year (recommended number of units 20)
Drawing - essential rules of drawing. Drawing square and oblong in grid, naming vertices and sides, pairs of adjacent sides. Circumference of square (oblong) - (only as a sum of sides' length, propedeutic). Sum and difference of segments' length. Multiple of segment's length. Drawing of triangle (if side lengths are given), naming vertices and sides. Measurement of triangle sides length with precision in cm and mm. Circumference of triangle - (only as a sum of sides length, propedeutic). Drawing arbitrary circle and disc with a given centre, circle and disc with given centre and radius. Properties of disc and circle. Converting units of length measurement. Converting of mixed units of length. Construction of solids from blocks according to a pattern (picture). Drawing plane of construction from blocks.

Compared with mathematical syllabus Mathematics for the 1st stage of basic school (1995) the scope of geometry knowledge has been reduced: drawing of perpendicular lines, right angle.

2 Geometry in elementary teacher training

Elementary teacher training in Slovakia is carried out in the two levels of tertiary education bachelor (Bc.) and magister (Mgr.).
Geometry is included in the following mathematical compulsory courses.

Bc. study
Introduction into mathematics
Communication language and elementary relations in geometry, Planimetry, Stereometry
Creating early mathematical concepts (mathematical-logical domain of preschool education)
Orientation in space and plain, Planar and three-dimensional geometric figures, Congruent representation, Points and lines, Measurement
Developing mathematical thinking in leisure-time pedagogy
Mathematical literacy - space and shape

Mgr. study
Geometry with didactics
Basic and derived concepts of Euclidean Geometry, Mutual positions of points, lines and planes in space, Planar geometric figures, Three dimensional geometric formations, Sets of points with given characteristics, Geometric representation. Congruence of geometric figures, Measurement of line, Measurement of planar formation
Methodology of teaching mathematics in primary school
Thematic area of Geometry and measurement in primary education

3 Testing probe into mathematical literacy - Space and Shape

One of the aims of the scientific project VEGA MMŠ a SAV 1/0192/08 Analýza matematickej prípravy študentov odboru Predškolskej a elementárnej pedagogiky z pohľadu rozvoja matematickej gramotnosti (Analysis of mathematical preparedness of the students of Pre-school and Elementary Education from the aspect of mathematical literacy) is a diagnostics of the level of mathematical literacy of the students - trainees (partial results were published in the papers (Gombár - Mokriš - Zeľová, 2009; Mokriš, 2010; Scholtzová, 2010; Zeľová - Gombár, 2010). One of the five tasks applied in testing was the task from the Space and Shape area (Mokriš, 2010). Testing of students was carried out at the outset of their study. Equal sample of respondents will be tested at the completion of their undergraduate training. In 2009/2010 academic year a testing probe of mathematical literacy was carried out

(Space and Shape area) among Bc. students in their second year of study and for comparison also among Mgr. students in their first year of study. Qualitative analysis of the students' solutions revealed several facts.

The first task was selected from the domain of stereometry - constructions from blocks. The standard of students' solutions was comparable in both groups. Mgr. students more often utilised their own graphic representations in their solutions.

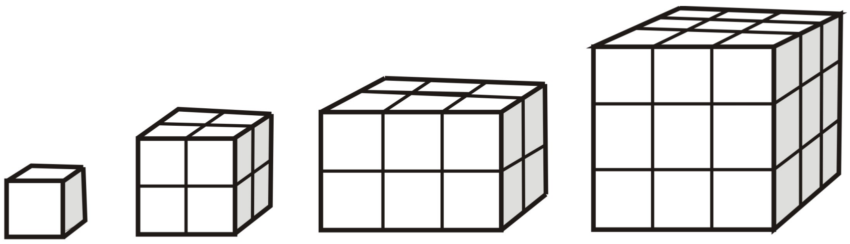
The second task included aspects as: properties of geometric figure (triangle), comparison of segments, mutual position of geometric figures, statement logic (conjunction of statements). The task was successfully solved by all Mgr. students and more than 60% of Bc. students.

The third task was stereometric again - pyramid and its characteristics. It's assignment had practical character. Half of the Bc. students did not attempt to solve the task. In Mgr. group it was only one student. The standard of solutions in both groups was very low.

The fourth task included measurement of planar figures (irregular) - circumference and area. The difference in the standard of solutions was most marked in this task. Bc. students applied intuitive approach to solution and estimate. Mgr. students in the majority of cases inserted figures into grid (Jordan theory of measure) and made attempts to systematise solving. The standard of their solutions was qualitatively higher.

Constructions from blocks

Susan is making constructions from small blocks (picture A) by gluing them together. First she glued together 8 blocks and made a construction depicted on the picture B. Then she made a construction depicted on the pictures C and D.



Picture 1 Picture 2 Picture 3 Picture 4

Question 1

How many blocks did Susan need to make a construction on the picture C? *Question 2*

How many blocks did Susan need to make a construction on the picture D? *Question 3*

Susan later realised that when she was making the construction on the picture D she used more blocks than it was needed. She could have made the construction hollow, yet it would have looked the same from the outside. What is a minimum number of blocks needed for such construction?

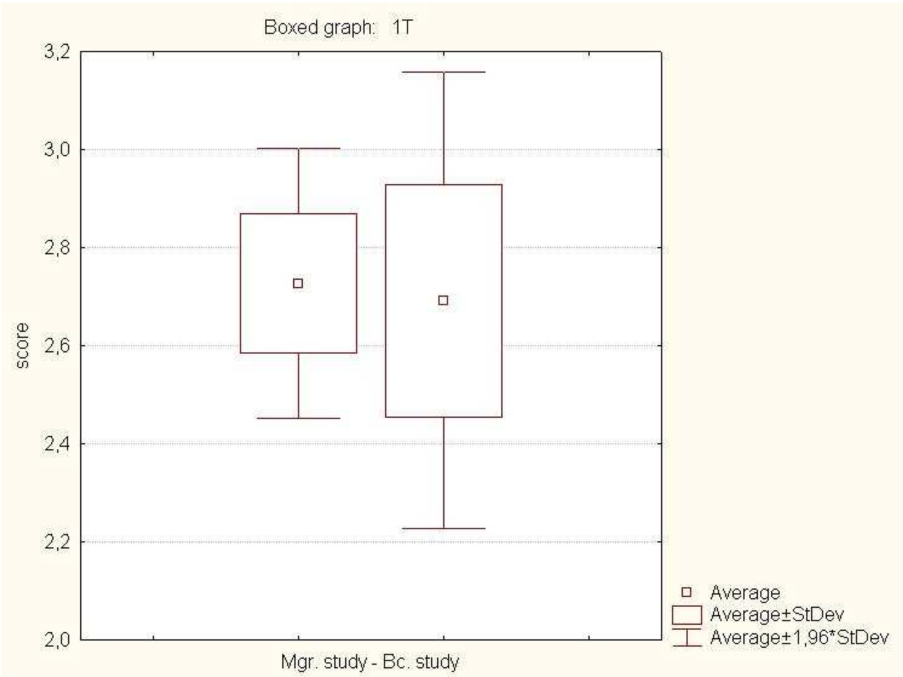
Question 4

Now Susan wants to make a construction which is 6 blocks long, 5 blocks wide and 4 blocks high. She wants to use the minimum number of blocks by making the largest possible hollow space inside the construction. How many blocks does she need for this construction?

Results achieved by students solving the task:

Descriptive statistics

	N valid	Mean	Median	Freq.	Min.	Max.	Disp.	St. dev.
1T-Bc.	107	2,439	3	55	0	4	0,645	0,803
1T-Mgr.	25	2,600	3	13	2	4	0,333	0,577



4 Conclusion

The testing probe into mathematical literacy in the Space and Shape area provided some information for improving the quality of elementary mathematical teacher training.

Observed negatives:

- insufficient graphic representation (visualisation) during solving geometry task,
- low ability to mathematicise real situation,

– success rate among Mgr. students was higher than in Bc. Students, however it cannot be stated that the difference is significant,

Observed positives include the following. Undergraduate mathematical teacher training:

– equalizes differences among individual students caused by completion of different types of secondary schools (different contents of mathematical syllabus),

– introduces new elements into mathematical abilities of students - effective methodological algorithms for task solving,

– influences students self-confidence, belief in one's own abilities - effort to solve a given task

References

[1] GOMBÁR, M.; MOKRIŠ, M.; ZEĽOVÁ, V. The Analysis of the Level of Mathematical Literacy of Elementary Students - Quantity Part. In *Mathematics from Primary Education View. The Conference Proceedings.* Banská Bystrica: Univerzita Mateja Bela, 2009. pp. 58 - 62. ISBN 978-80-8083-742-6.

[2] MOKRIŠ, M. Space and Shape – View of Mathematical Literacy of Undergraduate Students in the Field of Pre-school and Elementary Education. In *Matematika 4. Mathematical Education in a Context of Changes in Primary School. The Conference Proceedings. Acta Universitatis Palackianae Olomucensis. Facultas Paedagogica. Mathematica VII.* Olomouc: Palacky University, 2010. pp. 182 - 186. ISSN 0862-9765. ISBN 978-80-244-2511-5.

[3] SCHOLTZOVÁ, I. Some Aspects of Mathematical Literacy of Undergraduate Students in the Field of Pre-school and Elementary Education. In *Matematika 4. Mathematical Education in a Context of Changes in Primary School. The Conference Proceedings. Acta Universitatis Palackianae Olomucensis. Facultas Paedagogica. Mathematica VII.* Olomouc: Palacky University, 2010. pp. 271 - 276. ISSN 0862-9765. ISBN 978-80-244-2511-5.

[4] ZEĽOVÁ, V.; GOMBÁR, M. Change, Relationships, Dependence - Mathematical Literacy of Students of Pre-school and Elementary Pedagogy. In *Matematika 4. Mathematical Education in a Context of Changes in Primary School. The Conference Proceedings. Acta Universitatis Palackianae Olomucensis. Facultas Paedagogica. Mathematica VII.* Olomouc: Palacky University, 2010. pp. 321 - 325. ISSN 0862-9765. ISBN 978-80-244-2511-5.

[5] *Štátny vzdelávací program, Matematika, príloha ISCED 1, 2. upravená verzia pre 1. až 4. ročník ZŠ.* Bratislava: ŠPÚ, 2009.

[6] *Učebné osnovy matematiky pre I. stupeň základnej školy* (1995).

How to Change Students' (Grade 6) View of the Proof?

Zdenko Takáč

Catholic University in Ružomberok,
Faculty of Education, Department of Mathematics

Abstract. Results of an experimental study developed in elementary schools are presented. Relationships between intrinsic motivation to proving and special activity – engaging with so-called MRP tasks, are discussed. This article does not include complete answers of the research questions. Research results in detail are described in a more extensive article [4]. We turn our attention on the particular goal: to confirm an assumption that engaging with MRP tasks is reason for the change of students' (grade 6) view of the proof. An analysis of interviews with two students will be our supporting arguments for this assertion.

Keywords. Proof (in school mathematics), reasoning and proving, mathematics education, elementary school, grade 6, intrinsic motivation (to proving), MRP tasks, critical thinking (in mathematics).

The paper faces problem of motivation to proving in students and presents some results of an experimental study developed in primary and secondary schools. Theoretical framework and purpose for the study were described in [4]. We shall bring just brief description of MRP Tasks herein.

1 MRP tasks

MRP tasks (Motivation to Reasoning and Proving tasks) according to [3] are the tasks of following types:

Type 1 task that looks to have an easy solution, but after exhaustive dealing with problem, it has a different perhaps surprising solution.

Type 2 task that can be solved intuitively, but students are not sure of solution's correctness.

Type 3 task that has several possible solutions and students have to decide (and verify), which one is correct.

There is not sharp border of these three types of tasks. MRP tasks are not proving tasks, they do not explicitly require proof. Even dealing with most of them sense of proof does not appear. The aim of MRP tasks is "only" to weaken the solver's trust in solution on basis of guess or intuition without any verification. Furthermore, they can appear in any area of mathematics. Next are given three examples of MRP tasks and brief commentary.

MRP Task 1: John and Mary raced each other from a place A to a place B and back to A. Mary averaged 25 kmph cycling from A to B and 5 kmph walking back to A. John averaged 9 kmph running from A to B and back to A. Who has won?
MRP Task 2: A dog and a cat raced on the 100 m straight track there and back. The dog's jump is 3 m and the cat's jump is 2 m long. The cat makes 3 jumps while the dog makes 2 jumps. Who will win?
MRP Task 3: A knight is on square A1 of chessboard. Is it possible to repeatedly move the knight so that it will be just once on each square of chessboard and it will finish on square H8?

Note that an essential factor of Tasks 1 and 2 is in the surprise – students realize that even in seemingly simple situations can be confused if they had no verified their claim. A significant fact of Task 3 is that without verification (proof) students are not able to decide which solution is correct, but on the basis of relatively simple proof they "see it".

2 Data collection and analysis

We stated the research questions: We described link between children's dealing with MRP tasks and developing of their critical thinking in [3]. Is there a link between these two issues and increasing of children's intrinsic motivation to proving? Is there an impact of dealing with MRP tasks on increasing of intrinsic motivation to proving?

We are not going to answer the research questions completely in this paper. Research results in detail are described in a more extensive article [4]. There is conducted quantitative and qualitative analysis of students' performance on selected items. In this paper we turn our attention on the particular goal: to conduct an analysis of interviews with two students of grade 6. Our aim is to confirm an assumption that the change of students' view of the proof is a consequence of their engaging with MRP tasks.

The research was carried out by author in two elementary schools (21 classes were drawn as the subjects, grade 5-9) in Ruomberok, Slovakia. We will pay attention only to one of those classes - class grade 6, mark as 6EC3. Mathematical results of chosen class were average for the district. A mathematics teacher of the class, who had 8 years experience with the teaching at primary school, participated in observations.

Students of the class were asked to engage in one or two MRP tasks a month from September of 2007 to February of 2008. Teachers included these tasks in the classroom naturally. Students perceived it as a natural part of the learning process, not as a ”special activity”. They were tested twice: 1. pre-test – before dealing with MRP tasks (September of 2007) and 2. post-test – after dealing with MRP tasks (May of 2008). Pre-test and post-test were the same. According to the results we have selected 5 students and we have interviewed them.

The data for the article include the written responses of 23 students to two tasks from pre-test and two tasks from post-test, audio recordings and protocols of 5 interviews and teachers' reflections. The pre-test consisted of one “arithmetic task” and one “geometric task” (note that the post-test was the same).

Task 1 (pre/post-test, arithmetic task):
Topic: The sum of two even integers, the sum of two odd integers, the sum of even and odd integers.
1. Think about the topic and take down everything you will come to mind in connection with it.
2. Imagine that you are a teacher. Write a detailed preparation of lesson, which would have dealt with this topic (You may, but need not, use the knowledge of item 1).

Task 2 (final test, geometric task):
Topic: Quadrilateral whose vertices are midpoints of sides of rectangle.
1. Think about the topic and take down everything you will come to mind in connection with it.
2. Imagine that you are a teacher. Write a detailed preparation of lesson, which would have dealt with this topic (You may, but need not, use the knowledge of item 1).

We have formulated the tasks in the way that students had to solve the problem first (part 1 of task). Dealing with the arithmetic task they would realised that the sum of two even integers is an even number, the sum of two odd integers is an even integer and the sum of even and odd integers is an odd integer. Dealing with the geometric task they would realised that the quadrilateral with these properties is a rhombus. The students were under greater difficulties. Almost all students (21 of 23) were successful on the arithmetic tasks. Geometric task was somewhat more difficult, 18 of 23 students were successful. But for us it was not essential, whether students' solution was correct. We have paid attention whether they proved their claim or not and whether they proved it in the first part, in the second part or in the both parts of the task. Furthermore, we have not addressed the question whether the proof is correct, but whether students made effort to prove the claim. The term ”proof” is to be construed as any justification for their claim. But we have required the sense of a general view, not convincing by examples.

An analysis of the responses showed that 10 students of 23 tried on the proof in the first part of the task in the pre-test. All of them did it also on the post-test, but there were 5 students of the rest 13 who tried on the proof in the post-test, despite they did not make it in the pre-test. We interviewed these 5 students to discover why they revised their opinion. Next are given parts from two interviews and their analysis. The interviewer was the class teacher.

1 *Teacher:* Look at these two your answers. [Adam passed to John his start and post-tests.] You tried on the proof here, [Adam showed the post-test.] but you didn't try on the proof here. [Adam showed the pre-test.] Why?
2 *John:* Hmmm . . . [John did not say anything, he just red his two answers and he looked a little confused.]
3 *Teacher:* How did you proceed here? [Adam showed the post-test.]
4 *John:* I just put down what was on my mind.
5 *Teacher:* And what about September test?
6 *John:* It was the same.
7 *Teacher:* Yes it was. But why you tried on the proof in May, if you hadn't made it in September?
8 *John:* Hmmm . . . I hadn't? . . . I just put down [in May – post-test] what was on my mind and I wasn't thinking about what I've done in September . . . maybe . . . maybe something has changed . . . something in my mind . . . it was almost whole year.
9 *Teacher:* What do you mean – something has changed?
10 *John:* I . . . I don't know what has changed. I didn't know until now that I . . . that I have made it so different.
11 *Teacher:* But your two answers are almost the same. The only part that is different here is the proof. [Adam showed the post-test.]
12 *John:* Yes, I can see it. And isn't it an important difference?
13 *Teacher:* Is it an important for you?
14 *John:* I think it is. I don't know why I didn't try on the proof in September.

John was aware of the importance of the proof in mathematics (14, 12). This conclusion follows also from his surprise when he realized that he had not tried on the proof in September (2, 8: "I hadn't?"). Despite of this, he tried on the proof neither in the arithmetic task nor in the geometric task of the start test. He confirmed that in both cases he only wrote what he considers an important (4, 6). Simply put, John was not aware of the importance of the proof in September (before dealing with MRP tasks), but he was aware of this fact in May (after dealing with MRP tasks). Furthermore, in his own words his view has changed (8, "maybe something has changed, something in my mind"). Although he was confused (8, 10), his view of the proof was, indeed, changed after nine months. We claim that this is a consequence of his engaging with MRP tasks during this season.

1 *Teacher:* Why did you try on the proof here? [Adam put Samantha's post-test on the table.]
2 *Samantha:* I had to take down every important thing relating to given topic. I think this is an important.
3 *Teacher:* And why didn't you try on the proof here? [Adam put Samantha's pre-test on the table.] Do you remember? It was in September.
4 *Samantha:* Yes I do. Hmmm . . . I don't know. Maybe . . . , I forgot.
5 *Teacher:* Forgot?

6 *Samantha:* No, I couldn't forget. Hmmm ... I don't know. Maybe ... maybe I didn't consider the proof to be an important at that time.
7 *Teacher:* And do you consider the proof to be an important now?
8 *Samantha:* Yes, I do.
9 *Teacher:* Why?
10 *Samantha:* Because ... it is an important ...
11 *Teacher:* Why do you think?
12 *Samantha:* Because ... hmmm ... it is an important ... hmmm ... [40 seconds silence, Samantha is hard thinking] It is like the race, we all had thought that the runner could not win and then we proved that he won. [She referred to MRP Task 1 of this article.]
13 *Teacher:* Do you remember the task?
14 *Samantha:* Of course I do, I felt sure about my solution and I was wrong. I engaged in problem again at home.

Samantha's view of stuff has changed similarly like John's one. She was aware of the importance of the proof in mathematics in May (2), while she was not aware of this fact in September. Her assertions indicate change of her view of the problem (6, 8). We emphasize that, in this case, there appeared explicitly expressed link between addressing the MRP tasks to student and her opinion of the importance of the proof (12). Samantha clearly said that she considered the proof to be an important. In despite of this, she had major problems with giving reason for her claim (10, 12). After long thinking she argued by giving example of situation (MRP Task 1 of this article), in which she considered the proof to be crucial. Note that Samantha's class was engaged in this task in November.

3 Conclusion

Our conclusion is that the analysis of above described two sections of the interviews with John and Samantha discovered that the change of students' perspective on the proof and its significance was really caused by dealing with MRP tasks. Note that students who changed their mind from September pre-test to May post-test were interviewed. This is one of supporting arguments for the result of research: there exists the impact of dealing with MRP tasks (and developing of the critical thinking) on increasing of intrinsic motivation to proving. More supporting arguments for this result can be found in [4].

More research is needed to understand the way that teachers can participate in process of raising of students' intrinsic motivation to proving. In connection with MRP task new questions appeared. For example, has described increase of students' intrinsic motivation to proving long-term effect? We will search for the answer of this question in the near future. Also other questions are to be discussed. We find useful mainly matter of transition to a higher level of formality of proof. For example, described research outcome showed increase of students' intrinsic motivation to proving. But only without forcing students to the formal aspects of proof (students proved their own way). Question is whether this increasing is present also in proving on a higher level of formality.

The analysis presented in this article makes a contribution to the important topic of mathematics education - proving, particularly motivation to proving (see [2]). Research results suggest a link between motivation to proving, critical thinking and MRP tasks. As mentioned above, the transition to formal proof is crucial. But this transition is very facilitated by a good pre-training. If students at an early age feel the need for proof (with the appropriate formal level), the transition to a higher formal level is easier for them (see [1]). In particular, they can better keep a sense of need for proof and they will not come to believe that the proof should be done just to make a teacher happy.

References

[1] BACK, J.; PUMFREY, L. Primary proof? *Primary Mathematics*, 2005, 9(2), (http://nrich.maths.org/public/viewer.php?obj_id=2838).

[2] KOPKA, J. *Ako riešiť matematické problémy.* VERBUM, 2010. ISBN 978-80-8084-563-6.

[3] TAKÁČ, Z. Influence of MRP tasks on students' willingness to reasoning and proving. In Proof and Proving in Mathematics Education, ICMI Study 19, Volume 2, pp. 202-207, 2009, Taipei, Taiwan, ISBN 978-986-01-8210-1.

[4] TAKÁČ, Z. MRP tasks, critical thinking and intrinsic motivation to proving. *TMCS*, 2010, 8/1, 149-168, ISSN 1589 - 7389.

[5] ZELINA, M. *Aktivizácia a motivácia žiakov na vyučovaní.* Krajský pedagogický ústav Bratislava, 1989. ISBN 80-85185-09-1.

On Soccer Ball

Marián Trenkler[1]

Catholic University in Ružomberok,
Faculty of Education, Department of Mathematics

Abstract. A soccer ball is stitched together from 32 pieces of leather, 12 of which have the shape of a regular pentagon and 20 of a regular hexagon. In the lecture we examine the number and the shape of faces from which a soccer ball can be made. A part of the contribution is devoted to the study of metric properties of regular convex polytopes and the groups of their symmetries.

Keywords. Soccer ball, combinatorial properties, symmetries.

There are times when the whole world watches 22 young men chasing a ball on a soccer field, trying to get it into the competitor's goal.

In this contribution we shall study this small ball (685 – 690 in diameter and weighing between 425 and 435 grams) which, from time to time, gives many of us a hard time.

In 1863 the Football Association (also known as simply the FA) which approved the first Laws of the Game was established. There was no description of the soccer ball in these laws though. In 1872 the laws were revised, and the shape, size and weight of the soccer ball were set. With only minor changes, these regulations are used up till now. For official competitions there are more strict rules, e.g. the different diameters of the ball may not differ more then by 1,5 percent.

All this and much more (including an uncountable number of pictures) can be found on the Internet, hence we do not continue in this direction. Instead, we shall deal with the combinatorial properties of the soccer ball.

Let us start with the properties of the soccer ball which was used on FIFA (International Federation of Association Football) tournaments in the years 1970–2002. We shall explore its combinatorial properties and its symmetries.

This, the standard and the most known soccer ball, is stitched together from pieces of leather in the shape of 12 regular pentagons and 20 regular hexagons, which we shall call *faces*. Pairs of faces meet along *edges* and triples of faces meet

[1]This research is supported by KEGA project-463-035KU-4/2010

in *vertices*. It is natural to ask whether there exist different balls consisting of the same number of pentagons and hexagons with the property that three faces meet in each vertex.

The Steinitz theorem ([1], page 33.) states that a graph is the edge graph of a polyhedron if and only if it is a simple planar graph which is 3-connected. The soccer ball can be considered as a polyhedron. Hence, according to the Stenitz theorem, instead of the ball it suffices to consider a simple 3-connected planar graph (a graph is said to be k-connected if there does not exist a set of $k-1$ vertices whose removal disconnects the graph).

Let us state some properties of convex polyhedrons. If we denote by F, E and V the number faces, edges and vertices of a convex polyhedron, respectively, then the well-known Euler formula states [1]:

$$F - E + V = 2$$

Let us denote by F_k and V_k the number of k-gons and k-valent vertices (vertices of degree k). Then

$$F = \sum_{3\le k} F_k,\; V = \sum_{3\le k} V_k \quad \text{and} \quad 2E = \sum_{3\le k} kF_k = \sum_{3\le k} kV_k$$

By multiplying the Euler formula by six and using the above relations we obtain

$$(6F - 2E) + 2(3V - 2E) = 12 \tag{1}$$

$$6\sum_{3\le k} F_i - \sum_{3\le k} kF_k + 2\Big(\sum_{3\le k} 3V_k - \sum_{3\le k} kV_k\Big) = 12 \tag{2}$$

$$\sum_{3\le k}(6-k)F_k + 2\sum_{3\le k}(3-k)V_k = 12 \tag{3}$$

In a similar manner we obtain

$$\sum_{3\le k}(4-k)(F_k + V_k) = 8 \tag{4}$$

Relations (3) and (4) state the necessary conditions for the existence of a convex polyhedron. (Note, that in (3) there is no restriction on the number of hexagons and 3-valent vertices, and in (4) there is no restriction on the number of quadrangles and 4-valent vertices.) It follows

$$3F_3 + 2V_4 + F_5 \ge 12 \quad \text{and} \quad F_3 + V_3 \ge 8$$

These two relations imply that the degree of the faces and the vertices of a regular polyhedron can not exceed five (a *regular polyhedron* is a polyhedron whose faces are regular polygons of the same type which are assembled in the same way around each vertex). Moreover, if the faces are triangles, then all the vertices must be of degree three. The above relations give the necessary conditions for the existence of convex regular polyhedrons; there are five of them and they are often called the Platonic solids: tetrahedron, hexahedron (cube), octahedron, dodecahedron, icosahedron.

The ball which had been used in the last 30 years of the 20-th century consisted of pentagons and hexagons only and all its vertices were of degree three. The above formulas imply that such a ball must consist of exactly 12 pentagons but the number of hexagons is not given. Since this condition is necessary but not sufficient, it is natural to ask how many hexagons there can be. It follows from several papers published in the second half of the 20-th century that all the numbers except for the number one are possible (see [1], page 61). Figure 1 depicts the graph of a regular dodecahedron which in fact provides the basis for the soccer ball, as we shall see bellow.

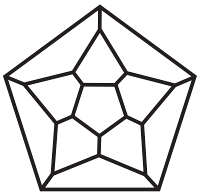


Figure 1. Dodecahedron

Given the planar graph of a polyhedron, it is possible to construct a new graph consisting of the same number of k-gons and new hexagons. This is illustrated on Figure 2. Dotted lines depict the original graph (the cube). On the left image the new graph is obtained by replacing all the original vertices by hexagons and the graph on the right is obtained by replacing all the original edges by hexagons. All the vertices of the new graphs are of degree three.

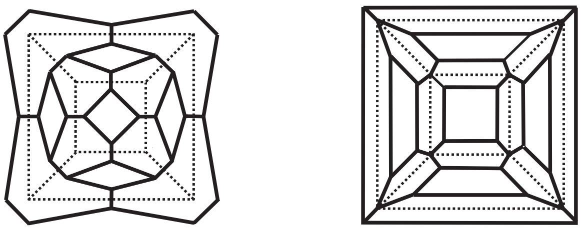


Figure 2. Transformations

Let us return to the soccer ball. It is constructed from the dodecahedron by replacing all the vertices by hexagons. Since the dodecahedron has 20 vertices, the soccer ball consists of 12 pentagons and 20 hexagons. By replacing the vertices by hexagons one more time, we obtain a ball consisting of 12 pentagons and 70 hexagons. By successive repetition we can obtain polyhedra consisting of 12 pentagons and a large number of hexagons.

By applying the second transformation on the dodecahedron, i.e. by replacing the edges by hexagons, we obtain a soccer ball consisting of 12 pentagons and 30 hexagons. If we apply this transformation on the cube, we obtain the ball used in volleyball. (Note: This ball can also be obtained from the octahedron by cutting-off the vertices.) Such balls were also used in football in the twenties of the last century.

Until the end of the 20th century, the ball was stitched together from pieces of leather. From practical reasons at most three pieces were stitched together in one point.

The first soccer balls in the 19th century were made of two digons stitched along the edges (such balls are still used in rugby). There were two vertices of degree two on the opposing sides and the ball was harder in these spots. All the balls used later were made so that at most three pieces of leather meet in one spot. Hence the balls were made of two 8-gons and 8 quadrangles, or other n-sided prism. Such balls were graph isomorphic with an n-sided polyhedron for $n = 8, 10, 12$.

The balls which were used later arised from the cube in four different ways. This is depicted on Figure 3 where the thick lines on each of the four pictures enclose one face of the cube.

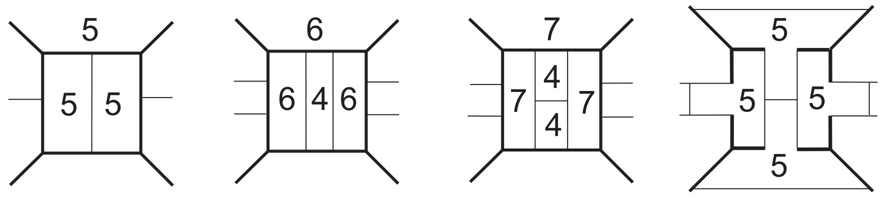


Figure 3. Constructions

Let us describe these constructions in detail.

1. We create a new vertex on each edge of the cube. We connect these vertices by six new edges so that on each face two new pentagons are created. Such a ball is graph isomorphic with the dodecahedron.

2. We create two new vertices on each edge and connect them by 12 edges so that on each face one new quadrangle and two new hexagons are created. Such a ball is graph isomorphic with the polyhedron obtained from the cube using the construction from Figure 2 on the right.

3. We start from the polyhedron constructed above - adding two vertices on each edge and 12 new edges. On each new edge we add a new vertex and connect each pair of the new vertices lying on the same face of the original cube by new a edge (thus creating the letter H). The faces of this polyhedron are 7-gons and quadrangles and their count is 24.

4. The polyhedron from point 3. contains 12 vertices incident with one quadrangle and two 7-gons. By removing the 6 edges connecting the pairs of these vertices we obtain a polyhedron which is graph isomorphic with the dodecahedron. Geometrically this polyhedron consists of 12 faces having the shape

pf the letter T. Such balls were used in the thirties of the 20th century. (Let us add that the valve was placed in one of the quadrangles, hence the two neighboring pentagons were transformed to hexagons.)

The above transformations can be done so that all the mirror planes of the cube are preserved. In such a case the resulting polyhedra have the same group of symmetries as the cube.

In the beginning of the 21st century synthetic materials replaced leather hence allowing for the creation of balls with various combinatorial structures. For the World Cup 2006 the ball called Teamgeist, containing six "biscuits " was used. The "biscuits" are quadrangles and the remaining faces are hexagons. This ball is graph isomorphic with the ball used in volleyball (depicted by full lines on Figure 2 on the right). For the world cup 2010 the ball called Jabulani was used. It is graph isomorphic with the tetrahedron whose vertices are replaced by triangles.

Observe the decorations on the soccer balls. The images "painted" on the ball usually reflect the structure of the faces. Some balls consisting of pentagons and hexagons have 5-starts painted on the pentagons; on others the hexagons are decorated by various ornaments. On some balls, the hexagons are replaced by triangles (in this case pairs of triangles "meet" in vertices of degree four). However, in each case there are two types of ornaments: 12 of them origin from pentagons and 20 from hexagons.

Finally let us consider metric properties of the soccer ball. Sometimes it is useful to know all the transformations which map the ball onto itself. These transformations are exactly the identities which map the dodecahedron onto intself. Carefully studying the soccer ball we can see that the dodecahedron, as well as the ideal ball, has 15 mirror planes. Each of these planes is given by a pair of "opposing" edges. By dividing the surface of the dodecahedron by the intersections of the planes with the surface, we obtain 120 elementary triangles. Since for each pair of triangles there exists exactly one identity mapping one onto the other, there are exactly 120 symmetries of the dodecahedron. Out of them 60 are direct and 60 opposite symmetries. However two identical shapes can not be identified in the 3-space. (More information on the symmetries of some polyhedra can be found in [2].)

Problem: In how many different (non-isomorphic) ways can we assign the numbers $1, 2, \ldots, 12$ to the pentagons of the soccer ball?

Solution: If the ball was fixed on a base then this number would be equal to the number of permutations of a 12-element set, i.e. 12!. However, there are 60 ditect symmetries (rotations) the dodecahedron and hence also the ball onto itself. Therefore among the 12! balls with assigned numbers there are groups of 60 balls whose numbers are the same when the ball is rotated in the proper way. Hence the number of different assignments is $\frac{12!}{60}$.

Regular polyhedra have three groups of symmetries. It is well known that the regular tetrahedron (cube and octahedron, respectively dodecahedron and icosahedron) has four (9, resp. 15) mirror planes and it is preserved by 12 (24, resp. 60) rotations. We can use these facts to solve several problems.

We can formulate a problem similar to the one above for the cube with one to six dots on each side. The cube has 9 mirror planes; three of them are given by the centers of the faces and six by the opposing edges. These planes divide its surface into 48 elementary triangles, hence there exist 48 identities mapping the cube onto itself. Only a half of these symmetries is direct hence there are 24 different assignment of the numbers $\frac{6!}{24}$ to the faces.

(Note: Dies used in table-top games have the additional property that the sum of the numbers on the opposing faces is seven. There are only two non-isomorphic dies like this and they are mirror symmetric.)

Problem: In how many different ways can we assign the numbers $1, 2, \dots, 8$ to the regular octahedron? How does this number change if we require the sum of the numbers on non-incident triangles to be the same?

The interested reader can formulate similar questions about different soccer balls.

References

[1] JUCOVIČ, E. *Konvexné mnohosteny.* Bratislava: Veda Bratislava, 1981.

[2] TRENKLER, M. On 4-valent 3-polytopes with a prescribed group of symmetries. *Graphs, Hypergraphs and Block systems, Zielona Gora*, 1976, 311–317.

Interpretations of Some Elementary Functions

Tomáš Zdráhal

Jan Evangelista Purkyně University in Ústí nad Labem,
Faculty of Education, Department of Mathematics and ICT

Abstract. Elementary functions, like the cosine, linear, exponential etc functions have interesting physical interpretations. To know them it is necessary first to define these functions by means of their characteristic functional equations. This article is intended as an attempt to make undergraduate university students acquainted with this process and reflection.

Keywords. D'Alembert's functional equation, Cauchy's functional equation, additive function.

Let us start with the cosine function $\cos\colon \mathbf{R} \to \mathbf{R}$ defined by the "cosine equation" (often called also d'Alembert's equation)

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{1}$$

where $f : \mathbf{R} \to \mathbf{R}$, for all $x, y \in \mathbf{R}$.
Further, consider a function that has the property:

— *the rate between the sum of function values on the ends and the function value in the middle of intervals from its domain is still the same if the intervals are of the same length.*

Thus if $[x-y, x+y]$ and $[t-y, t+y]$ are two intervals of the same length $2y$ and further $c(t)$ the value of a non-zero function c at the point t, then the function c satisfies the functional equation

$$\frac{c(x+y)+c(x-y)}{c(x)} = \frac{c(t+y)+c(t-y)}{c(t)}. \tag{2}$$

Setting in (2) $t := 0$ we get

$$\frac{c(x+y)+c(x-y)}{c(x)} = \frac{c(y)+c(-y)}{c(0)}. \tag{3}$$

Conversely, from the equation (3) by means of substitution $x := t$ follows

$$\frac{c(t+y)+c(t-y)}{c(t)} = \frac{c(y)+c(-y)}{c(0)}$$

and the substitution to (3) gives

$$\frac{c(x+y)+c(x-y)}{c(x)} = \frac{c(t+y)+c(t-y)}{c(t)},$$

which is the equation (2). The equations (2) and (3) are therefore equivalent. Consider the equation (3) for all possible real and search its solutions. If we restrict only to non-zero even functions we can transfer this equation (3) by substitution $f(t) := \frac{c(t)}{c(0)}$, $c(0) \neq 0$ to the d'Alembert's functional equation (1)

$$f(x+y)+f(x-y) = 2f(x)f(y); \tag{4}$$

it is easy to verify one of the solutions of this equation is the cosine function $\cos\colon \mathbf{R} \to \mathbf{R}$.

In fact, write (3) in the form (we assumed $c(-y) = c(y)$ for all $y \in \mathbf{R}$)

$$\frac{c(x+y)}{c(x)} - \frac{c(y)}{c(0)} = \frac{c(y)}{c(0)} - \frac{c(x-y)}{c(x)}.$$

Using the above substitution we get subsequently

$$\frac{f(x+y)c(0)}{f(x)c(0)} - \frac{f(y)c(0)}{c(0)} = \frac{f(y)c(0)}{c(0)} - \frac{f(x-y)c(0)}{f(x)c(0)},$$

$$\frac{f(x+y)-f(x)f(y)}{f(x)} = \frac{f(x)f(y)-f(x-y)}{f(x)},$$

$$f(x+y)+f(x-y) = 2f(x)f(y).$$

Show now we get (3) from the d'Alembert's equation (1). Write then (1) in the form

$$f(x+y)-f(x)f(y) = f(x)f(y)-f(x-y).$$

We have subsequently (for $c(0) \neq 0$)

$$\frac{f(x+y)}{f(x)} - f(y) = f(y) - \frac{f(x-y)}{f(x)},$$

$$\frac{f(x+y)}{f(x)} \cdot \frac{c(0)}{c(0)} - f(y) \cdot \frac{c(0)}{c(0)} = f(y) \cdot \frac{c(0)}{c(0)} - \frac{f(x-y)}{f(x)} \cdot \frac{c(0)}{c(0)}$$

and after the substitution $c(t) := f(t)c(0)$ we have

$$\frac{c(x+y)}{c(x)} - \frac{c(y)}{c(0)} = \frac{c(y)}{c(0)} - \frac{c(x-y)}{c(x)}. \tag{5}$$

Setting in (1) $y := 0$ we get $2f(x) = 2f(x)f(0)$. If we consider only non-zero solutions f implies from here that $f(0) = 1$. Setting $x := 0$ into (1) we have $f(y) + f(-y) =$

$2f(0)f(y) = 2f(y)$. It implies $f(y) = f(-y)$, i.e. f is a even function. Therefore also c is en even function as well and we can write (5) in the form

$$\frac{c(x+y)+c(x-y)}{c(x)} = \frac{c(y)+c(-y)}{c(0)},$$

which is the equation (3).

Let us yet briefly mention about similar interpretations for a linear and exponential functions, respectively.

Let a function have the property:

— *the difference between the function values on the ends and on the beginnings of intervals from its domain is still the same if the intervals are of the same length.* Thus if $[t, t+y]$ and $[x, x+y]$ are two intervals of the same length y and further $l(t)$ the value of a function l at the point t, then the function l satisfies the functional equation

$$l(x+y) - l(x) = l(t+y) - l(t). \tag{6}$$

Setting in (6) $t := 0$, we get

$$l(x+y) - l(x) = l(y) - l(0). \tag{7}$$

Conversely, putting in (7) $x := t$ we have $l(t+y) - l(t) = l(y) - l(0)$ and together with (7) $l(x+y) - l(x) = l(t+y) - l(t)$, i.e. the equation (6). Thus the equations (6) and (7) are equivalent. Search for solutions of equation (7). First let us transfer this equation by means of the substitution

$$f(t) := l(t) - l(0) \tag{8}$$

to the Cauchy's functional equation

$$f(x+y) = f(x) + f(y); \tag{9}$$

it is easy to verify the continuous solution is of the form $f(x) = ax$. Let us recall the solution of this equation (9) is called an additive function. Putting thus (8) into (7) we get

$$f(x+y) + l(0) - f(x) - l(0) = f(y) + l(0) - l(0),$$

that is just the Cauchy's functional equation

$$f(x+y) = f(x) + f(y).$$

Show that by substitution (8) we get conversely from the Cauchy's equation

(9) equation (7). First let us write (9) in the form

$$f(x+y) + l(0) - (f(x) + l(0)) = f(y) + l(0) - (f(0) + l(0))$$

(we use the fact that from (9) implies (by putting $x = y := 0$) $f(0) = 2f(0)$, i.e. $f(0) = 0$). By the substitution $l(x) := f(x) + l(0)$ we get

$$l(x+y) - l(x) = l(y) - l(0),$$

which is equation (7).

Finally, consider now a function with the following property:

— *the rate between the function values on the ends and on the beginnings of intervals from its domain is still the same if the intervals are of the same length.*

Thus if $[t, t+y]$ and $[x, x+y]$ are two intervals of the same length y and further $e(t)$ the value of a non-zero function e at the point t, then the function e satisfies the functional equation

$$\frac{e(x+y)}{e(x)} = \frac{e(t+y)}{e(t)}. \tag{10}$$

Putting in (10) $t := 0$ we get

$$\frac{e(x+y)}{e(x)} = \frac{e(y)}{e(0)}. \tag{11}$$

Conversely, from equation (11) by substitution $x := t$ we get $\frac{e(t+y)}{e(t)} = \frac{e(y)}{e(0)}$ and putting this into (11) we get $\frac{e(x+y)}{e(x)} = \frac{e(t+y)}{e(t)}$, i.e. equation (11). Thus equations (10) and (11) are equivalent.

Consider equation (11) for all possible real x, y and search for its solutions. By substitution

$$f(x) := \frac{e(x)}{e(0)}, \quad e(0) \neq 0$$

we get the equation

$$f(x+y) = f(x)f(y), \tag{12}$$

that could be solved easily:

$$\frac{f(x+y)}{f(x)} = \frac{f(x+y)e(0)}{f(x)e(0)} = \frac{e(x+y)}{e(x)} = \frac{e(y)}{e(0)}$$

$$= \frac{f(y)e(0)}{e(0)} = f(y).$$

Show yet that equation (12) can be transfered to equation (11) vice-versa: For $f(x) \neq 0$, $e(0) \neq 0$ we get consequently

$$f(x+y) = f(x)f(y),$$

$$\frac{f(x+y)}{f(x)}\frac{e(0)}{e(0)} = \frac{f(y)}{f(0)}\frac{e(0)}{e(0)}.$$

Here we used the fact $f(0) = 1$. Indeed, from (12) for $x = y := 0$ implies $f(0) = f(0)f(0)$. If $f(0) = 0$ was valid then we would get from (12) for $y := 0$ the equality $f(x) = f(x) \cdot f(0)$, thus f should be a zero function. Therefore must be $f(0) = 1$. Further put $e(x) := f(x)e(0)$. We get then

$$\frac{e(x+y)}{e(x)} = \frac{e(y)}{e(0)},$$

which is equation (11). As for solutions of equation (12) the following theorem is true:

Theorem 1. *Let $f: \mathbf{R} \to \mathbf{R}$ be a solution of equation* (12)*, i.e. the equation*

$$f(x+y) = f(x)f(y) \tag{13}$$

for all $x, y \in \mathbf{R}$. Then either $f = 0$ or an additive function exists (i.e. a solution of the Cauchy's functional equation (9)*) $a: \mathbf{R} \to \mathbf{R}$ such that*

$$f = \exp a. \tag{14}$$

Conversely, any function of the form (14)*, where $a: \mathbf{R} \to \mathbf{R}$ is an additive function satisfies equation* (12)*.*

References

[1] ACZÉL, J. *Lectures on functional equations and their applications.* New York – San Francisco – London: Academic press, 1966.

[2] ZDRÁHAL, T. Looking for functions. *Selected Topics from Math. Ed.*, 1996, Volume 1, Number 4, 36–42.

[3] ZDRÁHAL, T. *Functional equations and elementary functions.* Acta Universitatis Purkynianae, Studia Mathematica. Ústi nad Labem: Univerzita J. E. Purkyně, 2009.