NONSTANDARD TASK AS A TOOL FOR DEVELOPING MATHEMATICAL ACTIVITY

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Abstract:
This article is about nonstandard task as a tool for developing mathematical activity.
Keywords: mathematical knowledge
MESC: C70, D40

The essence of mathematics, understood as a field of human activity, is to solve tasks. Mathematics is seen here as a science which developed in the course of working on tasks, subjectively new definitions and theorems. In the same time, mathematical knowledge and skills have been playing a more and more important role in our daily lives, while solving tasks has been the backbone of teaching mathematics at every level. Having in mind the goal of preparing students, in the course of the educational process, to living in the surrounding reality, one should emphasize the tasks which allow the pursuit of general objectives of mathematical instruction, i.e. those developing skills and attitudes necessary to a modern person, regardless of his or her field of activity. This goal may be reached by assigning non-standard tasks to students.

By standard tasks we mean those that meet the following criteria:

- there is sufficient information to obtain an unambiguous solution and at the same time there are no redundant data,
- the content of the task does not lead to contradiction,
- the content of the task is adequate, by which it is meant that the questions are closely connected to the data, the task relates to the real life, its conditions are sufficiently precise, and the task may be subject to arithmetic mathematization.

Negating any of these features leads to a non-standard task (see Gleichgewicht, 1988). Solving this type of tasks develops, among others, intellectual attitudes evidenced by logical, creative, and independent thinking as well as by overcoming difficulties, and can improve the ability to analyze the content of the task and understanding of the global structure of the task. As Polya notes, in mathematics itself, skills are more important than knowledge. What is skill in mathematics? It is the ability to solve problems, and not just typical tasks but those that require independent judgment, judgment ability, originality, creativity (Polya, 1975).

In contemporary books for students at different levels of education, there are very few non-standard tasks. We analyzed a few typical publications which are used by a significant number of students. The following publications were analyzed:

1. for primary school students: ``Matematyka z plusem’’, ``Matematyka 2001”, ``Matematyka krok po kroku”;
2. for high school students: ``Matematyka z plusem", ``Matematyka 2001", ``Matematyka wokół

nas".

In the analyzed material, the majority of tasks are those with redundant (non-contradictory) data. In total, 792 such tasks were noticed at the primary level, and 1523 at the high school level. In most of them, data presented in the table, diagram, or chart constitute an integral part of the task. There are quite a lot of tasks with ambiguous solutions. In total, 202 such tasks were noticed at the primary school level, and 350 at the junior high school level. At the high school level 30 tasks had contradictory content. At primary level, there were 12 such tasks. Very few tasks were not fit for arithmetic mathematization. At the high school level, one such task was reported, at the primary level 22 tasks. Such a small number of non-standard tasks is worrying. If the general objectives of teaching mathematics are to be achieved, it must be ensured that there are more non-standard tasks.

It should also be noted that non-standard tasks are rare in the first three grades of elementary school. These are some tasks with redundant data, tasks that are not subject to arithmetic mathematization, and arithmetic tasks with ambiguous solutions. In (Dabrowski, 2007), the results of a study in which third grade pupils solved tasks are presented. A typical was simple task solved correctly by 86% of the students, while a simple task with redundant data by only 52%. Similar problems can be observed at the level of the second and third stage of education.

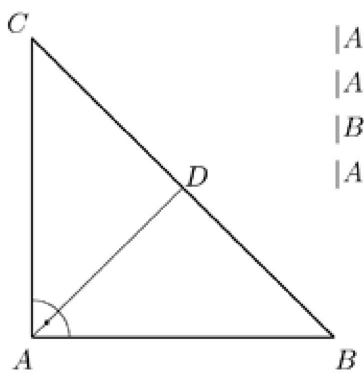
In the paper (Major, 2012) the attitude of students in relation to non-standard tasks is described. In particular, it was observed that the majority of respondents working on the tasks with redundant (non-contradictory) data showed no reflection on the inadequate amount of data present in the problem. A number of students solved the task correctly without giving any comment on either the structure of the task or the course of their reasoning.

Below, we present a task which was proposed to 15 high-school students who like math. This task includes conflicting data.

Task 1. *In the right triangle ABC the sides lengths are:* $2\sqrt{3}$, $2\sqrt{3}$ *and* $2\sqrt{6}$. *The length of the line segment joining the right angle's vertex to the midpoint of the hypotenuse is equal to* $\sqrt{5}$. *Find the area of the triangle and of its circumcircle.*

During the study, the students solved the task and wrote the solutions on a piece of paper. Solving this task the students received: Solution 1-4 students, solution 2-5 students, solution 3-3 students and solution 4-3 students.

Task 1. Solution by student U1.



$|AB| = 2\sqrt{3}$ $\quad P_{\triangle} = ?$

$|AC| = 2\sqrt{3}$ $\quad P_{\circ} = ?$

$|BC| = 2\sqrt{6}$

$|AD| = \sqrt{5}$

$$P_\triangle = \frac{1}{2} \cdot 2\sqrt{3} \cdot 2\sqrt{3},$$
$$P_\triangle = 6\,\mathrm{j}^2.$$

To find the area of a circle we need to calculate the radius. It is a right triangle so the radius is equal to half the length of the hypotenuse:

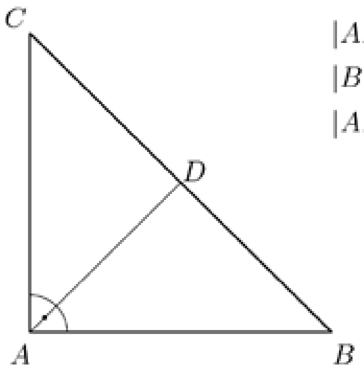
$$r = \frac{1}{2}|CB| = \frac{1}{2} \cdot 2\sqrt{6},$$
$$r = \sqrt{6}.$$

Therefore

$$P_\circ = \pi r^2$$
$$P_\circ = 6\pi\,\mathrm{j}^2.$$

Answer. The area of the triangle is equal to $6\,\mathrm{j}^2$; the area of the circle is equal to $6\pi\,\mathrm{j}^2$.

Task 1. Solution by student U2



$|AB| = |AC| = 2\sqrt{3}$ $\quad P_\triangle = ?$

$|BC| = 2\sqrt{6}$ $\quad P_\circ = ?$

$|AD| = \sqrt{5}$

$$P_\triangle = \frac{1}{2}|AB|^2,$$
$$P_\triangle = \frac{1}{2} \cdot (2\sqrt{3})^2,$$

$$P_\triangle = 6\,\mathrm{j}^2.$$

Point D is the center of the circle circumscribed about the triangle so the length of the segment AD is equal to the radius of the circle:

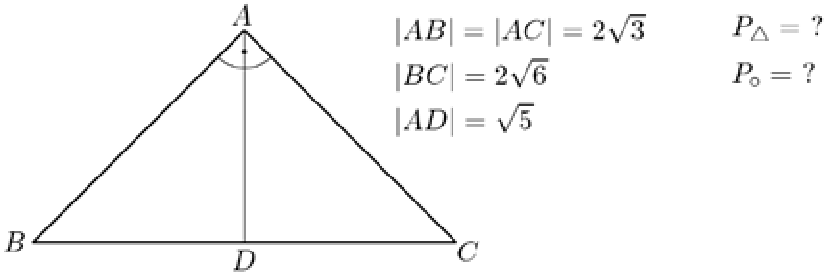
$$r = |AD| = \sqrt{5},$$
$$r = \sqrt{5}.$$

Hence

$$P_\circ = \pi r^2,$$
$$P_\circ = 5\pi\,\mathrm{j}^2.$$

Answer. The area of the triangle is equal to $6\,\mathrm{j}^2$; the area of the circle is equal to $5\pi\,\mathrm{j}^2$.

Task 1. Solution by student U3



Triangle ABC is an isosceles right triangle; thus, segment AD joining the vertex of the right angle to the midpoint of the hypotenuse is the altitude (since it is perpendicular to the segment BC)

$$P_{\triangle} = \frac{1}{2}|BC||AD|,$$

$$P_{\triangle} = \frac{1}{2}\cdot(2\sqrt{6})\cdot\sqrt{5},$$

$$P_{\triangle} = \sqrt{30}\,\mathrm{j}^2.$$

Segment AD has a length equal to the radius of the circle because D is the midpoint of segment BC and in a right triangle the circumcenter is the midpoint of its hypotenuse. Therefore

$$r = |AD| = \sqrt{5},$$

$$r = \sqrt{5}.$$

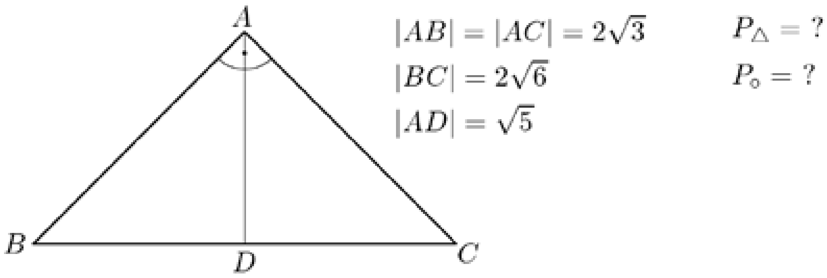
Hence

$$P_0 = \pi r^2,$$

$$P_0 = 5\pi\,\mathrm{j}^2.$$

Answer. The area of the triangle is equal to $\sqrt{30}\,\mathrm{j}^2$; the area of the circle is equal to $5\pi\,\mathrm{j}^2$.

Task 1. Solution by student U4



$$P_{\triangle} = \frac{1}{2}|BC||AD|,$$
$$P_{\triangle} = \frac{1}{2}\cdot(2\sqrt{6})\cdot\sqrt{5},$$
$$P_{\triangle} = \sqrt{30}\,\mathrm{j}^2.$$

In a right triangle, the radius of its circumcircle is equal to half the hypotenuse BC. Hence

$$r = \frac{1}{2}|BC| = \frac{1}{2}\cdot 2\sqrt{6},$$
$$r = \sqrt{6}.$$

Therefore

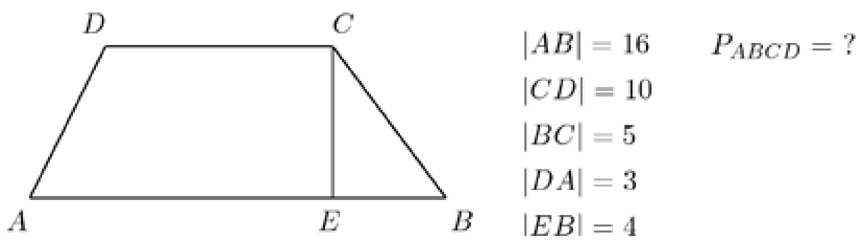
$$P_0 = \pi r^2,$$
$$P_0 = 6\pi\,\mathrm{j}^2.$$

Answer. The area of the triangle is equal to $\sqrt{30}\,\mathrm{j}^2$, the area of the circle is equal to $6\pi\,\mathrm{j}^2$.

After solving the task, each student received three solutions of the problem, different than their own. Having read the solutions, the students stated that the solutions analyzed contained errors. None of the respondents did not see the essence of the problem, despite indications such as: Analyze the content of the tasks, look at the data present in the task. After showing students the contradiction of data in the problem and discussing tasks, they were asked to solve the following task.

Task 2. In the trapezoid $ABCD$, segment AB is the longer base and $|AB| = 16$, segment CD is the shorter base and $|CD| = 10$. Also $|BC| = 5$ and $|DA| = 3$. The foot of the altitude from vertex C to the longer base is marked as point E. Segment EB's length is equal to 4. Find the area of trapezoid $ABCD$.

Task 2. Solution by student U1



The area of the trapezoid is given by the formula

$$P_{ABCD} = \frac{1}{2}(|CD| + |AB|)\cdot|CE|.$$

From the Pythagorean theorem for triangle ECB, we have:

$$|BC|^2 = |CE|^2 + |EB|^2,$$

$$|CE|^2 = |BC|^2 - |EB|^2,$$

$$|CE|^2 = 9,$$

$$|CE| = 3.$$

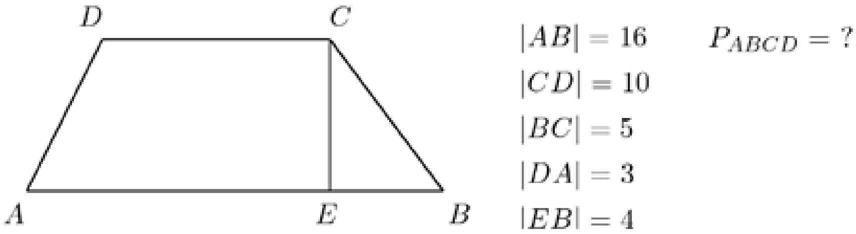
Therefore

$$P_{ABCD} = \frac{1}{2}(10+16)\times 3,$$

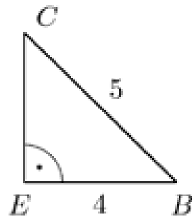
$$P_{ABCD} = 39\,\mathrm{j}^2.$$

Answer: The area of trapezoid $ABCD$ is equal to $39\,\mathrm{j}^2$.

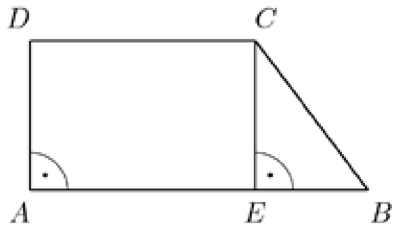
Task 2. Solution by student U2



EBC is a Pythagorean triangle hence $|CE| = 3$.



We know that $|DA| = 3$ so $|DA| = |CD|$. Hence, $ABCD$ is a right trapezoid.



Since $|CD|=10$, we obtain $|AE|=10$. Thus we have:

$$P_{ABCD}=\frac{1}{2}(|CD|+|AE|+|EB|)\times|CE|,$$

$$P_{ABCD}=\frac{1}{2}(10+10+4)\times 3,$$

$$P_{ABCD}=36(\mathrm{j}^2).$$

Answer: The area of trapezoid $ABCD$ is equal to $36(\mathrm{j}^2)$.

Students presented solutions, like with the previous task. It was only when they received a solution other than their own (where the results were different from those received by the particular person), students would say: It’s a strange task again. Students participating in the study stated even that these tasks are not real mathematical problems.

When working on the tasks, the respondents demonstrated significant difficulties as to the division of the task, a problem mentioned by S. Turnau among others (1990). These difficulties appear in the phase which G. Polya describes as the phase of understanding the task. The students did not consider all relations between the data. They did not notice contradiction. When shown various solutions (using different data), students looked for errors in the reasoning presented. Only during the work on the second task, students began to check if the data are not contradictory. The results indicate an urgent need for more non-standard tasks in students’ education at each level. In the everyday life, it happens very frequently that, trying to solve a problem, we select data and this requires ensuring that they are not contradictory.

References

GLEICHGEWICHT B., Arytmetyczne zadania tekstowe w klasach I-III, WSiP, Warszawa 1988.

MAJOR J. The attitudes of participants in the process of solving mathematical tasks with insufficient data, w: Scientific Issues. Mathematica IV, ed.: Billich M., Catholic University, Ružomberok 2012, 109-115.

POLYA G. , Odkrycie matematyczne, WNT, Warszawa, 1975

DĄBROWSKI M., Pozwólmy dzieciom myśleć, CKE, Warszawa, 2007.

TURNAU S., Wykłady o nauczaniu matematyki, PWN, Warszawa, 1990.

DOBROWOLSKA H., Jucewicz M., Zarzycki P., Matematyka z plusem kl IV, GWO, 2011.

DOBROWOLSKA H., Jucewicz M., Zarzycki P, Matematyka z plusem kl V, GWO, 2011.

DOBROWOLSKA H., Demby A., Matematyka z plusem kl VI, GWO, 2011.

CHODNICKI J., Matematyka 2001, kl IV, WSiP, 2012.

Matematyka 2001, kl. V, prac zb., WsiP, 2011.

Matematyka 2001, kl. VI, prac zb., WsiP, 2011.

Matematyka z plusem kl. I, II, III GWO, 2011.

Matematyka 2001, kl. I, II, III prac zb. WSiP. 2011.

Matematyka wokół nas kl. I, II , III, WSiP, 2011.

Forms of Education of Gifted Pupil in Mathematics at Primary School

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Abstract:
The report presents findings of practice as they are used in various forms of education of integrated gifted pupil. The attention is focused on the primary school pupil which shows the mathematical talent and which is integrated in regular class at school. The report especially deals with various forms of mathematics teaching. The advantages and disadvantages of teaching methods are discussed below.
Key words: Mathematics, gifted, primary education, organizational form.
MESC: D20, D40

1 Introduction

Gifted people bring the innovation, new ideas and new solutions to problems. In the world and in our country there is growing interest in the development of potential of gifted pupils. The attention is paid to the development since pre-primary and primary education. There are a number of myths about the education of gifted pupils for example that the gifted pupils don´t need any special care or they can handle this situation by themselves and that they should adapt to the rest of the class . Every child is unique and has a right to develop his or her potential. Clark (2009) formulated A Declaration of the Educational Rights of the Gifted Child from which, in the context of organizational forms of teaching, we state below:

It is the right of a gifted child:

- to engage in appropriate educational experiences even when other children of the grade level or age are unable to profit from the experience,
- to be grouped and to interact with other gifted children for some part of the learning experience in order to be understood, engaged, and challenged,
- to be taught rather than be used as a tutor or teaching assistant for the major part of the school day.

In the text we will devote to the organization of teaching gifted pupil at school. Not in special amateur clubs after school or in extracurricular activities. When planning the educational program for gifted students the basic principles should be respected. The principles including by Sisk (1987, p. 72) that *„education for the gifted should reflect an inclusive rather than exclusive attitude"*. Gifted pupils have special needs, they have a right to progress themselves by own pace. Various measures are being implemented to reflect these needs. Arrangements for gifted are very often grouped into these categories: enrichment, acceleration and special grouping.

2 Arrangements for gifted

Enrichment. „*It is the right of a gifted child to be taught the concepts that the child does not yet know instead of relearning old concepts that the child has already shown evidence of mastering.“ „It is the right of a gifted child to pursue interests that are beyond the ability of age peers, are outside of the grade level curriculum, or involve areas as yet unexplored or unknown.“* (Clark, 2009) The aim of the enriching educational offer is the broadening and deepening of the curriculum. But it is not just about getting new information and facts but mainly about discovering new relationships and connections in the broad context. It is about learning new procedures for solving problems, the use of higher level thinking and acquiring of learning skills.

Acceleration. *“It is the right of a gifted child to learn faster than age peers and to have that pace of learning respected and provided for*.“ (Clark, 2009) Acceleration programs allow to gifted students faster progress during the school education. It is a premature start of the school attendance, skipping classes, thickening of teaching content, for example from two school years to one.

It should be given a special attention to the acceleration in mathematically gifted education, as Gardner (1999) states: most mathematicians create their major work before the twenty-fifth or thirtieth year of their life. But it is a mistake to approach the acceleration in the mathematical education of very young children if there were not yet fully exploited the possibilities of enriching educational processes. The teacher should offer to pupils a good balanced learning provision with a support in social and individual endeavours. A scale of variety forms by Sisk (1987): Regular classroom (with cluster/pull-out), individualized classroom (cluster/pull-out), special classes, special schools.

The organizational form of teaching is one of the external factors of the education. It is one of the prerequisites for successful teaching characterizes the framework in which the learning takes place (Průcha, 2009). We state the organizational forms of teaching due to the position of a gifted pupil, which is integrated into the regular class with whom we met during the research of training mathematically gifted students at primary school.

2.1 Gifted student without a special care

It is not given a special care to the gifted pupil. He or she receives the same tasks as the other pupils in the class. The tasks are extremely easy for him or her. The pupil solves the tasks very quickly and then for a relatively long time he has no meaningful activity. This is repeated several times during the teaching lesson. Although the care of gifted pupils is anchored in binding documents still this situation persists in many schools. As we verify during the structured observation (with a ten-seconds interval) of gifted pupils during maths the pupil´s effort to work changed in undesirable behaviour thanks to inappropriate content, methods and forms. It was awarded a number of similar tasks to all pupils in the class in which they applied only multiply two-digit numbers. Not only that the gifted student was bored and did not work but after a short time he began to search for activity by himself – he disturbed classmates by talking and nudging. Although there was a strange person in the class, the pupil forgot at her for a moment and an undesirable behaviour was observed – physical attacks of classmates which the teacher never noticed. The outputs from structured observations are evident from the graph Nr. 1 and graph Nr. 2. During the lesson the pupil quickly solves the presented tasks and spent the remaining time by wandering in thoughts or he disturbed classmates. It seems that the time share the pupil worked 47 % is high. It is important to note that the tasks did not develop the pupil, they were too easy for him.

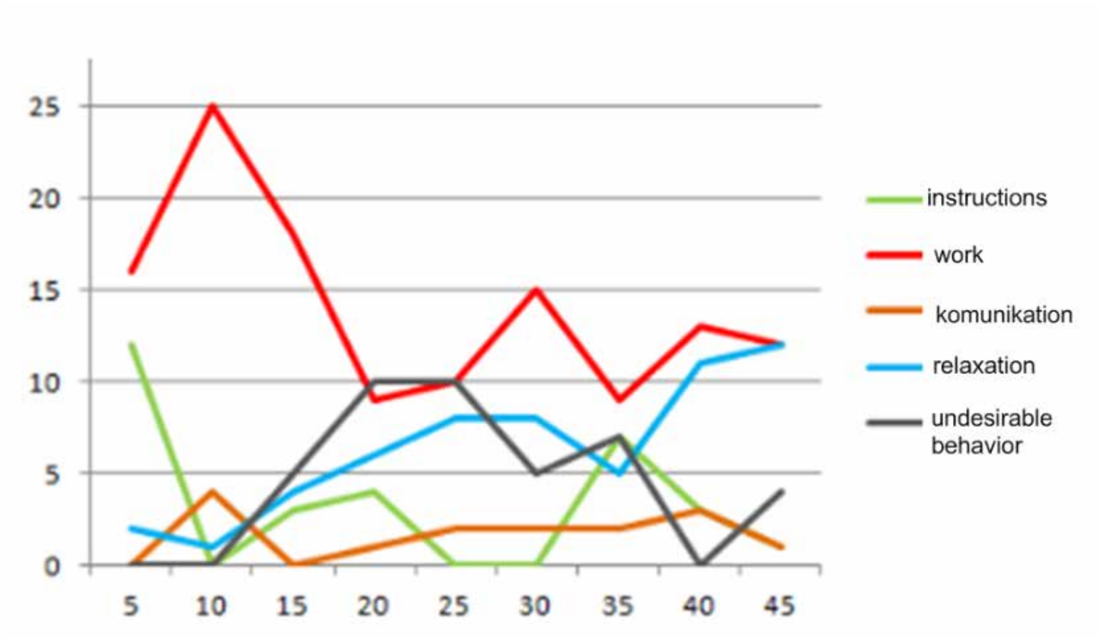

Fig. 1 The Time share of the pupil activities in five minute segments during the lesson

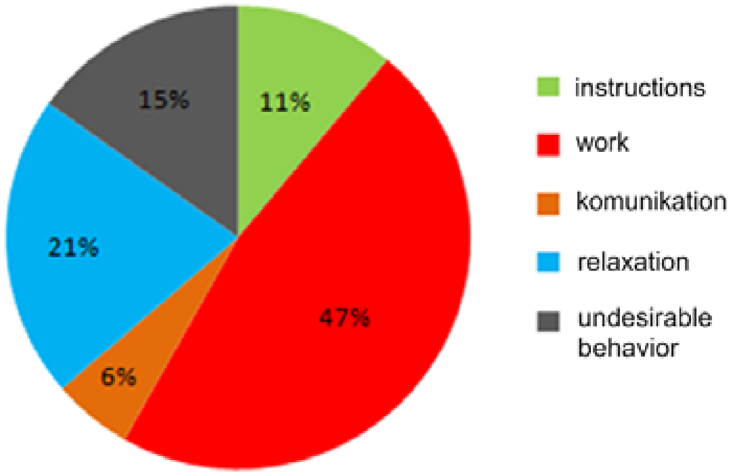

Fig. 2 The time share of the pupil during lesson

2.2 Gifted student participating in a differentiated classroom

When applying the differentiated instruction in the regular class the activities are divided by current cognitive levels of pupils. There is a tiered education when all pupils work on one topic but the teacher presents different roles depending on the individual learning needs of individual pupils. This form of learning in regular class puts high demands on the teacher's work. In practice it is irregularly used by experienced teachers with years of experience. More flexible form of work is ability grouping where pupils receive different instructions according to current ability. They do not necessarily work on the same topic.

2.3 Gifted pupil receives tasks in addition during the teaching

A gifted pupil is done with the tasks prior to classmates. He receives the tasks in addition from the teacher during the lesson. He usually receives more tasks from the textbooks, from the chapter that is being discussed. Sometimes they are also assigned the "star-tasks" of the special parts of textbook. The teacher is in individual contact with the pupil during a lesson. The teacher enter him ongoing tasks and regularly checks his work in a workbook. The pupil participates along with others in the class in some parts of the teaching lesson. Sometimes he discontinues a work on his specific task and returns to work together. During our research observations we recorded that the application of this organizational form to challenge gifted pupil is often impaired by interruptions of work on his task by the teacher. Gifted pupil has an intense desire to know, an intense need to complete the task, to find a solution. That is why he (or she) does not hear directions or is discomforted by interruption.

2.4 Gifted student works individually in the time block

A gifted pupil works individually on a special task or set of tasks during the teaching lesson. He works in the block and he is not disturbed from the work. The tasks are challenging, but they are prepared preferably addressed to solve independently by the pupil without the support of teachers. Tasks that are attractive to challenge the gifted pupil, having the following characteristics: a reasonable difficulty, an interesting thematic content, a realistic context, precise instructions without substantive and formal errors. The solutions of the problem are to be a higher level of thinking, balance issues requiring convergent and divergent thinking. For primary school pupils is appropriate to assign tasks of the activity character with handling material. It is beneficial if the pupil can give himself the feedback through teaching aids or by comparing his solutions with the pattern. Also the gifted students need to practice various math skills. However, they do not need to practice as much as other peers. It is not appropriate to assign a large number of routine tasks, which appeal to memory and drill.

The example of suitable tasks: *The buns are baked on a pan. Some buns are at the edge of pans, another in the middle. How many buns can be on a pan if the number of cakes at the edge is equal to the number cakes in the middle. (someone likes "fried", the other not).* (Blažková, 2012)

2.5 Two gifted pupils in the class

If there are two gifted pupils in the class it is appropriate to implement teaching „work in pairs“. In practice, we have encountered such a situation in two classes. Two pupils (schoolboys and schoolgirls) showed the general intellectual talent or specific mathematical talent. At the initiative of teacher the talent was also diagnosed by the experts of pedagogical- psychological counseling. At the prompt of the teacher these gifted students sit together in the classroom, either for the all part of the lesson or only for a part of it. They get tasks different from other students in the class. However, this is a pseudo-type of the group work. The pupils solve these tasks individually.

They only discuss the solutions of the tasks together. It is desirable to prepare tasks suitable to a work in pairs which require cooperation. It is possible to give students school supplies to a school desk or to assemble such a task that the cooperation of students is inevitable. As an example of such a task we present the author's task – Melichar (2013):

The „scratch out“ game: Two players compete with each other. Each player selects two numbers of a group of numbers and cross them out. Then the players calculate the mathematical *product of these numbers place them in the appropriate position in the table and write down the number of points they earned. The competition continues with non scratched numbers. The winner is the one who gets more points.*

55	21	46	57	33	19	71	47	18
70	61	68	47	24	39	48	34	
59	52	23	65	11	49	38		

0 - 999	1 000 - 1 999	2 000 - 2 999	3 000 - 3 999	4 000 - 5 000
1 point	2 points	3 points	2 points	1 point

Pupil	Points											Sum
A												
B												

3 Conclusion

A gifted student needs to be accepted with their specific educational needs and also to incorporate into peer groups and prevent its exclusion. Schoolmates should include a gifted pupil. Teachers should incorporate the teaching learning strategies allowing accommodate the unique learning styles of the gifted pupils even in the primary education. The choice of appropriate organizational forms is crucial in education of the gifted pupil which is integrated into the regular class. So much desired differentiation and individualization in teaching mathematics is conditional upon the skills of teachers to choose the right educational resources – mathematical tasks - in accordance with the organizational forms. Training of mathematically gifted pupils which are integrated into the regular classroom is one of the challenges for contemporary mathematics education.

References

BLAŽKOVÁ, Růžena. Přístupy nadaného žáka k řešení úloh. In Učíme nadané žáky. Ostrava: Ostravská univerzita, 2012, s. 260-263. ISBN 978-80-7464-104-6.
CLARK, Barbara. Growing up gifted: developing the potential of children at home and at school. 3rd ed. Columbus: Merrill Pub. Co., c1988, xiii, 674 p. ISBN 06-752-0832-7.
GARDNER, Howard. Dimenze myšlení. Praha: Portál, 1999, 398 s. ISBN 80-717-8279-3.
MELICHAR, Jan. Odhad výsledků. In: MALINOVÁ, Dagmar. Nadání je třeba rozvíjet - vzdělávání nadaných v základní škole: Metodická příručka a soubor pracovních karet. 2013. (V tisku)
PRŮCHA, Jan. Pedagogická encyklopedie. Praha: Portál, 2009. 936 s. ISBN 978-80-7367-546-2.
SISK, Dorothy A. Creative teaching of the gifted. New York: McGraw-Hill, c1987, 370 p. ISBN 00-705-7701-3.

NUMERAL BLOCKS AS AN EDUCATIONAL TOOL FOR DEVELOPMENT OF MATHEMATICAL THINKING IN PRESCHOOL PREPARATION

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Abstract:
In the framework of the project ESF We Manage to Do It Together – CZ.1.07.1.-2.00/08.0105 the author proposed a teaching aid that helps to develop mathematical thinking. This aid was tested in practice in preparatory classes in which is practised the educational program for children from poor social-cultural environment. The abstract describes the aid and methodology of its use.
Key words: Project, aid, preparatory class, development of mathematical thinking.
MESC: F10, F30

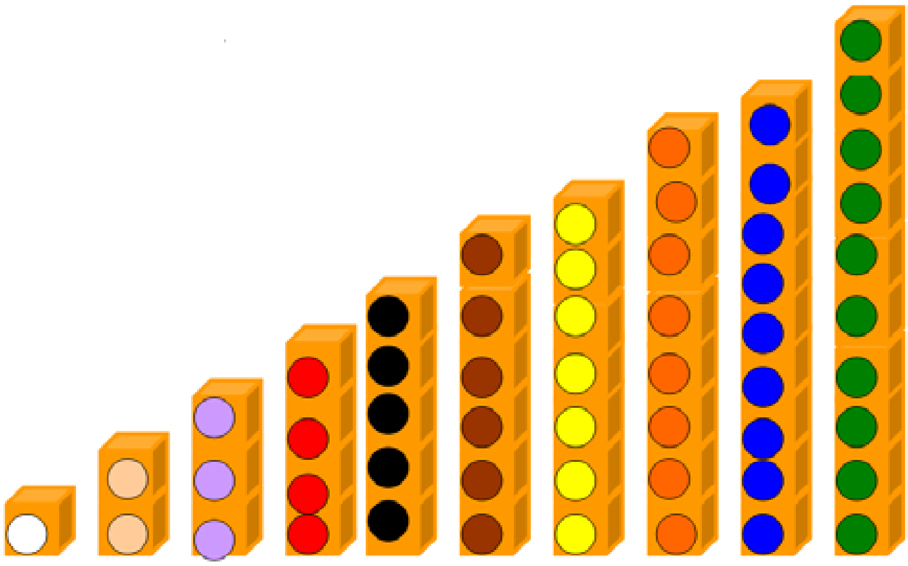
Within the project of European Social Fund „We Manage to Do It Together“ – CZ.1.07.1.-2.00/08.0105 the author in cooperation with company Komárek from Staré Město produced an educational tool for preparatory classes in which are children from poor social-cultural environment. The tool was called Numeral Blocks. This tool originate in similar tool known as Cuiseair´s Blocks or Morozovov´s Colourful Blocks (Klocki Kolorowe). Original blocks differ in colour and length and there are not marked values. Due the absence of marked values it is more difficult for children to work with them. Block with marked value serve as better model for natural number.

Let´s see the tool and hot to work with the tool:

Title of the tool: Numeral Blocks

Description of the tool:

10 different blocks:



Each numeral block is characterized by its length and colour. Length of the block is marked on it by colour spots. Size of the block is 1 x 1 unit of length and its height is from 1 to 10 units of length based on the number that the block represents. Colours characterize certain multiple.

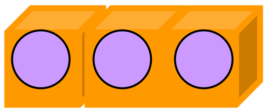
Set of numeral blocks contains 10 white blocks (1), 5 light orange blocks (2), 4 light blue blocks (3), 3 red blocks (4), 2 black blocks (5), 2 brown blocks (6), 2 yellow blocks (7), 2 dark orange blocks (8), 2 dark blue blocks (9) a 2 green blocks (10).

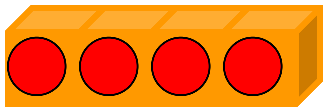
How to use blocks in preparatory class?

1 Training of comprehension of natural numbers from 1 to 10 (numeracy in range to 10):

Number is noted down by numerals and pronounced as ordinal number.
Each child including teacher lay out on desk set of 34 numeral blocks.

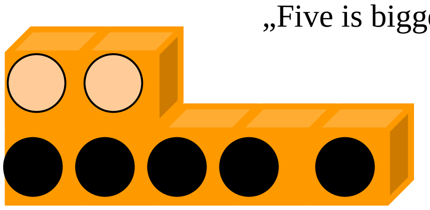
a) Teacher pronounces various numbers and children are pointing at block with corresponding length and colour, e.g. Teacher says “three”.....children show:

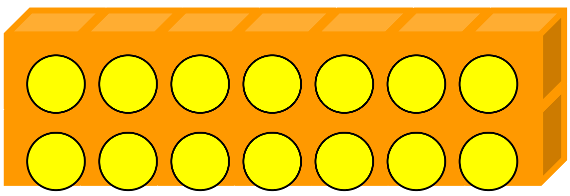


b) Teacher shows:  children say „four“. Etc.

c) Children form couples. One child is playing the teacher. One child is pronouncing numbers and the other child is pointing at corresponding block. Than children switch their roles.

d) Children are learning to compare numbers. Teacher takes two different blocks, children compare them and decide whether numbers represented by the blocks are equal or whether is one number bigger or smaller than the other one. Children are expressing themselves verbally.

e.g. :  „Five is bigger than two.“, „Two is smaller than five“.

 „Seven is equal to seven“.

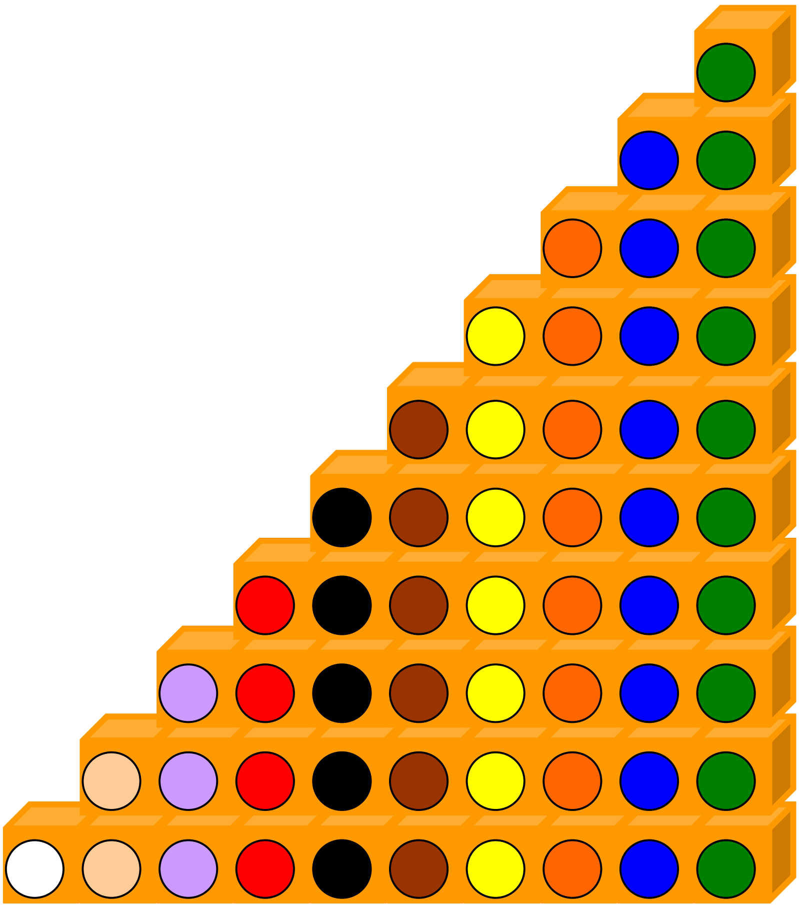
e) Game called “The King“. Children form two groups with the same number of members. They stand in a queue. One child stands before the queue and holds in hand non-transparent sack with set of 10 blocks of various length and colours. Teacher says e.g. „five“. Children that are standing at the beginning of their queue/row run and try to find in the sack corresponding block without looking into the sack. Child that finds the correct block as a first moves up into next round of the game.... up to the final round. In final round compete winners of previous rounds. The winner is “The King”. Blocks are always put back into the sack.

f) „Stairs“ - Children create stairs and say rhymes that help them to learn to count „one, two, three, four, five, six, seven, eight, nine, ten“.

“One little, two little, three little Indians, Four little, five little, six little Indians, Seven little, eight little, nine little Indians, Ten little Indian boys.

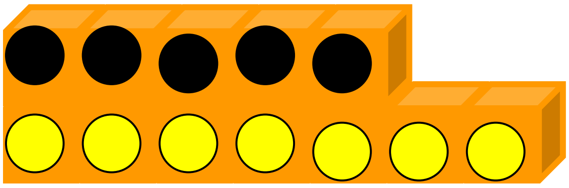
“One for sorrow, Two for joy, Three for a girl, Four for a boy, Five for silver, Six for gold, Seven for a secret never to be told. Eight for a wish, Nine for a kiss, Ten for a bird that you won't want to miss.”

Etc.



2 Addition and subtraction in domain up to ten

a) Teacher lay on a table two blocks of different colours.
E.g.:



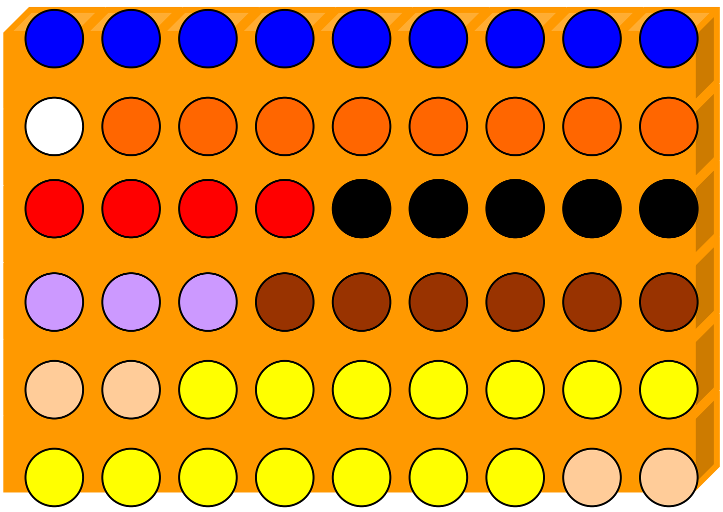
Children do the same. Children are trying to add additional block to reach the same length. They add light orange block of the length “two”. Children say: “black block and light orange block are equal to yellow block” or they can say „five and two equal to seven“.

CAUTION! Children do not note any numbers down. Everything is performed orally.

b) Children take various couples of blocks and search the third relevant block. We can ask children to solve the problem: two blocks of the same length. Children come to the conclusion that they do not need to add any block. This way we come to number “zero”.

c) We can use a die. Children throw a die two times. Value of both throw determines blocks children should take.

d) „Carpets“ – children choose e.g. block “nine” and form all possible couples of blocks relevant to the length “nine”.



„one and eight is nine“

„four and five is nine“

„three and six is nine“

„two and seven is nine“

„seven and two is nine“

PROGRAMS FOR VISUALIZATION SOME MATHEMATICS OPERATIONS

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Abstract:
A visualization and creation of models along with their application to demonstrate and study various objects and phenomena of the real world play an important role in the demonstrative education. In this paper, the author considers a possible utilization of computer programs for visualization in the teaching. She present several publicly available applications as well as the programs created as a part of students' Bachelor and Master theses at the Faculty of education of Catholic University in Ružomberok.
Key words: visualization, programs, matrices, algorithm
MESC: N60, N80

1 Introduction

Visualization is a method for the visual interpretation of data, which uses the advantages of human perception in the analysis. This process transforms symbolic data into data that can be seen, and a process allows researchers to display their abstract simulations and calculations. Visualization offers "see invisible" and contributes to the detection of new information in many branches. An appropriate visualization allows students to understand the presented problem better and more quickly. It also makes it easier to understand causes and consequences of the changes. The visualization also provides an option to highlight or suppress some important features, which is convenient during the first contact of the students with the environment.

Visualization algorithms and data is no simple process. Design must be easily understandable and attractive to the user. For the animation is necessary to choose appropriate timing. Interactive visualization must, actively drawn the user into the process. Proper color combination is in computer graphics and in visualization a very important aspect

We can use various programming languages (Turbo Pascal, Delphi, C++, HTML, Macromedia Flash, Java) for creating visualization applications. Author must realize to the resulting product designed, with such data will work and whether it will be interactive or not.

2 VIMO

VIMO program (Krišš,2013) designed and implemented is designed primarily for students of mathematics, or computer science as a tool to study the subject of algebra, discrete mathematics, or Practice of Programming 1.

Programming language Pascal, in which the program was created, has sixteen colors, which may seem low, but too much color visualization could be disturbing. Square matrices are for visual processing operations matrix most suitable. (LEHOTSKÝ,M., KRAKOVSKÝ, R., POVAŽAN, J.,2006)The program VIMO works default with square matrices. The program VIMO offers visualization mirroring, transposition, rank, calculation determinant and three basic operations: sum (difference) of two matrices and the conjunction of two matrices.

Main menu consists of items Uploading, Operations at 1 matrix, Operations at 2 matrices, About program and items End. The user moves on this menu with the cursor up and down with arrow keys and confirms selection with the Enter key.

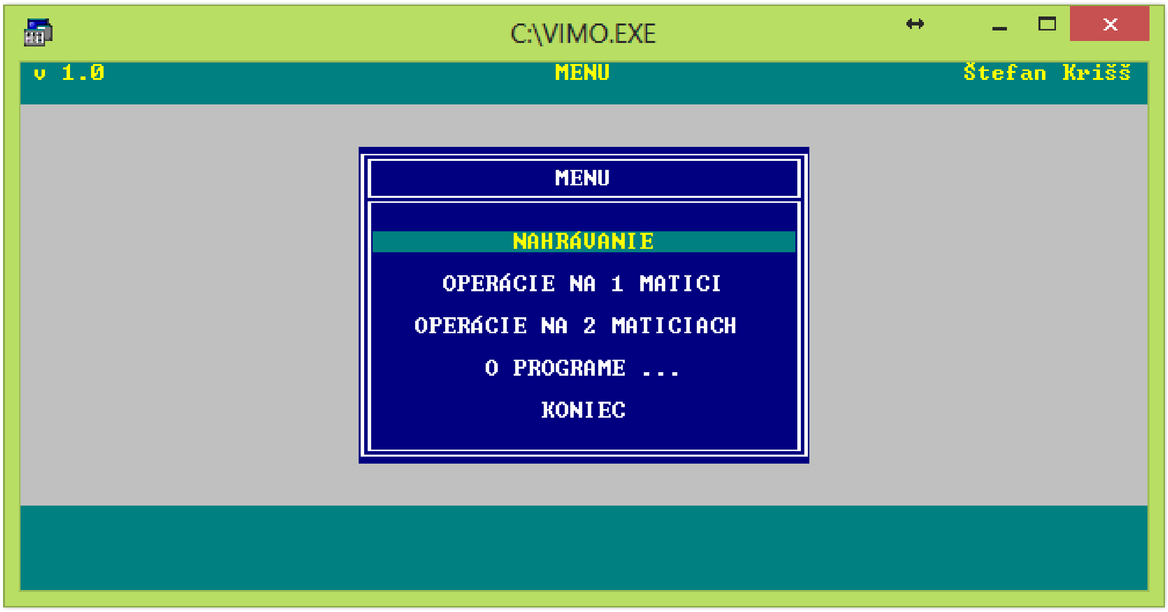

Fig. 1 Main menu

The user can load the matrix automatically, or will specify all the elements. Randomize function that assigns to each element a random number from the interval <0,9>, starts to automatically fill matrix. This automatic filling takes place quickly and at the end shows the information that both matrices were loaded. The user must press any key to continue.

If the user specifies all the elements of the matrix, use the Upload manually. He can only enter numbers from the interval <0, 9>, to simplify the abbreviature of matrix. If a user accidentally or intentionally entered a number outside the default interval, program wait to load a right numbers.

A problem rewritten matrix is treated, before entering the recording menu. If a user clicks on an item recording matrix in the main menu accidentally or intentionally, when both matrices are already filled, a warning will be displayed with ask, if the user wishes to rewrite the matrix. Entry Operations at 1 matrix contains items Mirroring, Transposition, Rank, Determinant and item Back. Visualization of the calculation is displayed after clicking on the item

Columns and rows of the matrix may be a mirror. The program allows first alternative of this operation. Mirroring column matrix is effective for imaging. Mirroring algorithm can better understand on first alternative. Program VIMO highlighted column in the original matrix after starting the algorithm, and then it prints the appropriate position in a mirrored array. Dividing line in the middle of the matrices can be considered axis of symmetry.

The transposition procedure ensures a defined exchange rows and columns. Procedure highlighted column in the original matrix and its elements appear on the screen like row transposed matrix.

Number of non-zero rows matrix determines its rank. The procedure adjusts the matrix to a triangular shape with the help of elementary line adjustments. The program prints the resulting rank by number of nonzero rows. Visualization line editing is very difficult, so the program prints the resulting matrix. This warning is displayed in the program under the matrix. (Figure 2) This procedure displays a list of regular and singular matrix. Matrix is singular, if the rank of matrix is not equal to the degree of the adjusted matrix. If the number is the same as the degree matrix, is a regular.

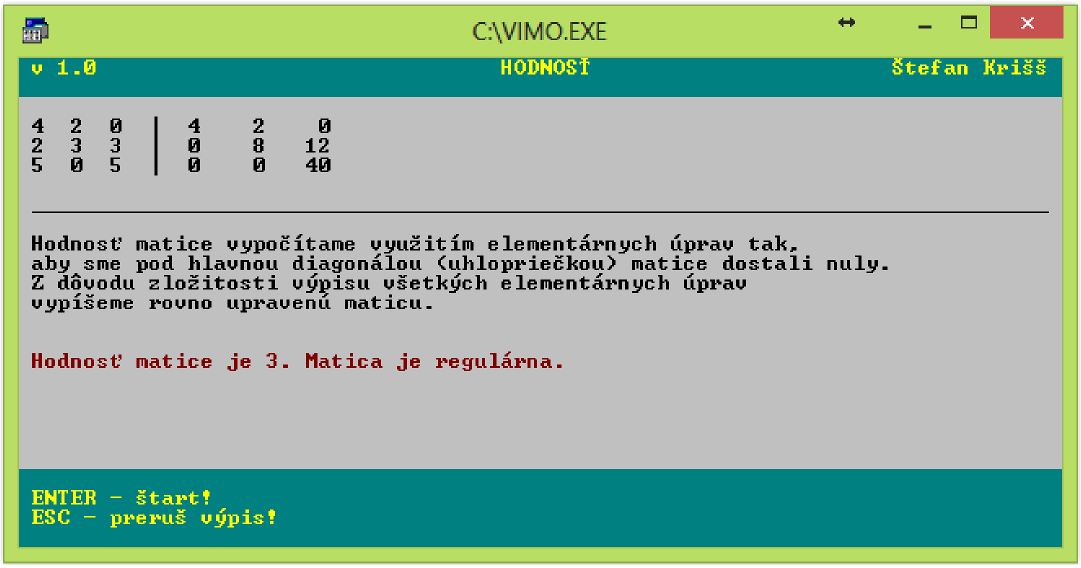

Fig. 2 The rank of matrix

The user can choose from the item Operations on 2 matrices of three basic matrix operations of sum, difference and multiplication of two matrices.

Operations of sum and difference have a common procedure. Procedure has an input parameter of type char name sign. The sign determines how the operation goes. The whole algorithm is started, when you press Enter. The calculation of each element of the resulting matrix is represented by four steps:

- element at position (1,1) in the matrix A is highlighted,
- the sign "+", is highlight,
- element at position (1,1) in the matrix B is highlighted,
- element at position (1,1) in the resulting matrix is displayed.

The whole cycle is repeated until all elements of the resulting matrix is displayed.

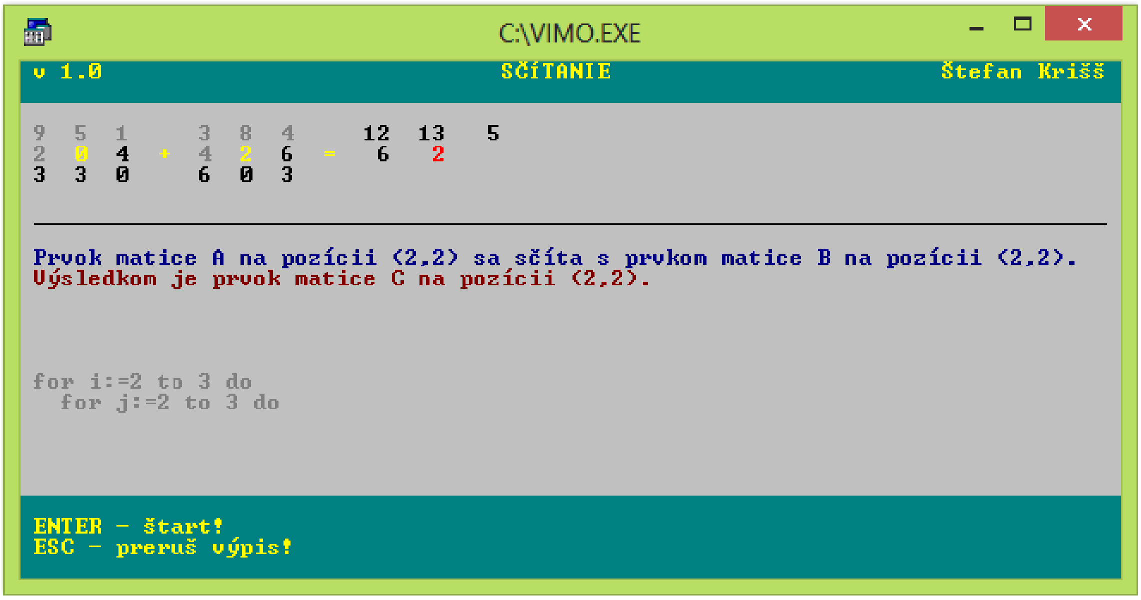

Fig. 3 Sum of two matrices

Visualize difference operation is the same, but its input parameter is the sign "-". elements that the program have already processed, are grayed out (Figure 3), that the user understand in which step algorithm is

Procedure conjunctions multiply two matrices with each other. The algorithm has three steps. Row in the first matrix and column in the second matrix are paint at the beginning. Cycle, which elements of these vectors multiplied and summed with each other in pairs, will start in the second step. This process is displayed under their matrixes on the line. The resulting element of the final matrix lists at the end. Elements that have already been processed are painted gray. We can see in the figure 4, that under the matrix is a note, that verbally describes the work with the elements. Pseudocode cycle for is below the note on the processing element

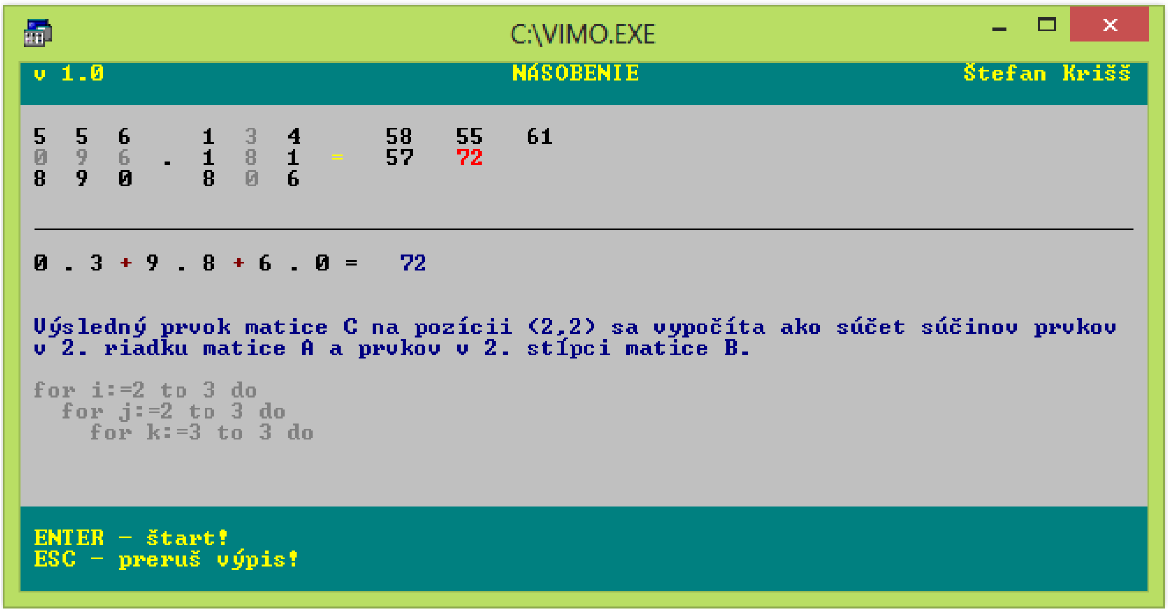

Fig. 4: Multiply of two matrices

3 Visualization program for theory of graph

As an example of the visualization in the graph theory, we present a program for the visualization of selected algorithms, which has been developed in the children's programming language Imagine. The program was created as a Master thesis of a student at the Pedagogical Faculty of Catholic University. (Daubnerová,2011). The program allows displaying algorithms for minimum skeleton, minimum path in a graph. The environment contains buttons for the instructions, a button for clearing, and two buttons for selecting the algorithm for drawing a skeleton of the graph (fig. 5). A button ALGORITHMS is used to list the individual steps of algorithms that were used during the calculation. Each step contains a short description of what does it comprise. We can select Kruskal´s algorithm or Prim´s algorithm to find the minimum skeleton graph. Consequently, we can write a matrix into the command line. This matrix contains the weights for the individual edges of the graph.

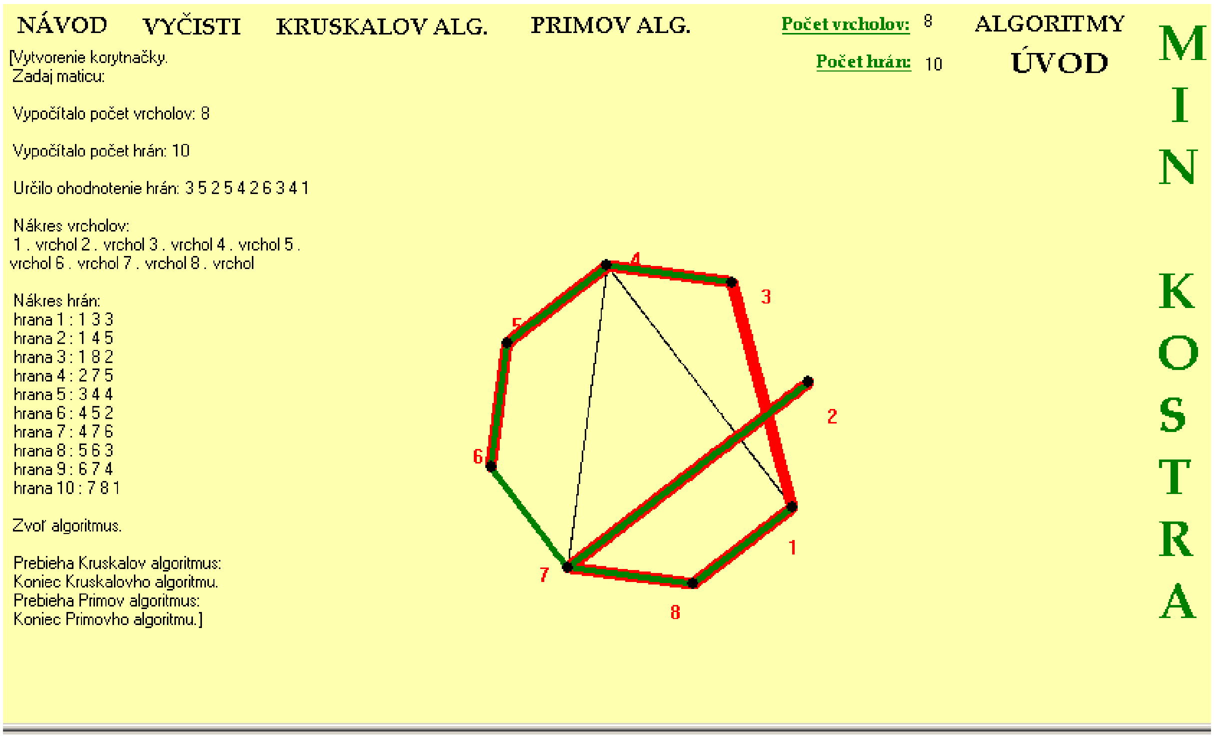

Fig. 5 Visualization algorithm of skeleton of graph

The program displays a number of vertices and edges, a list of the weights of edges in a consecutive order, and order of plotting of vertices and edges. For each edge, a starting and ending vertex along with its weight are displayed. The program gradually plots a minimum skeleton of a given graph. We can display the minimum skeleton obtained separately by both algorithms in a single chart. We can also see the summary of individual steps of the algorithms. The second algorithm is the one for finding the minimum path in the graph. The program draws the graph and minimum path between two vertices (fig. 6). We can also get the individual steps of the Floyd´s algorithm.

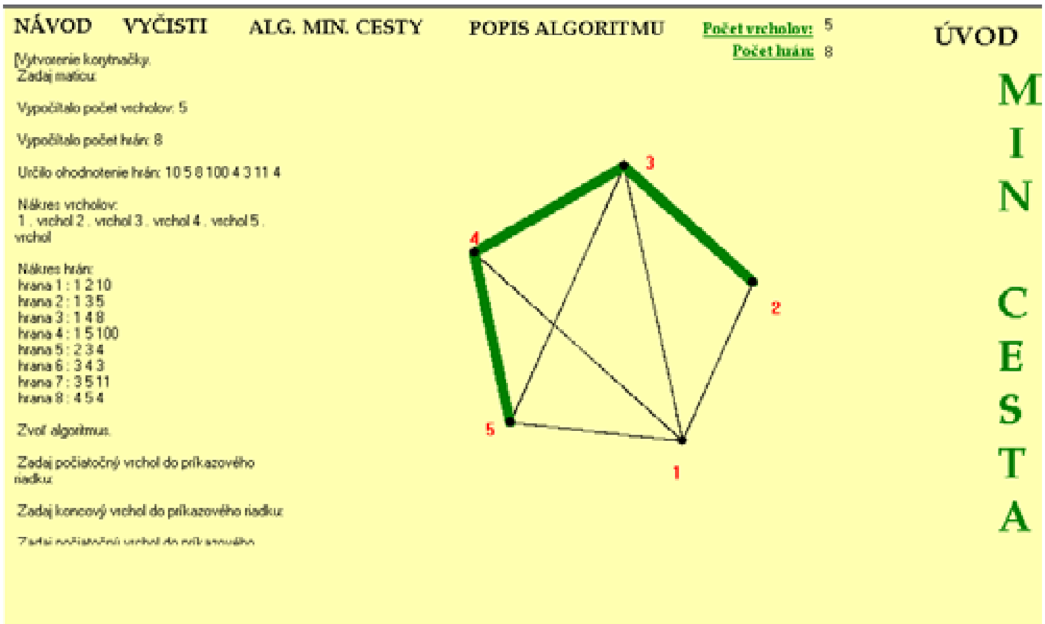

Fig. 6 Minimum path algorithm visualization from vertices 2 to vertices 5

Another program Simple Graph for visual processing of the algorithms has been created in the programming language Delphi 7.0. (Prílepková, 2012) The students can take advantage of the clear link to knowledge of discrete mathematics and theoretical computer science. Main menu consists of several parts: Djikstra´s algorithm, Ford-Fulkerson´s algorithm and the most likely path. We can also choose the plotting speed of the respective algorithm. In this program we can create an arbitrary graph, which can be used to demonstrate a specific algorithm and procedure of its calculation. A toolbar consists of ten tools that can be used during a process of drawing the graph (creation of a new graph, loading graph from the file, save the graph to a file, insert vertex into a graph, insert edges into a graph, set the vertex - beginning, set the vertex - end, set the vertex - default, delete objects marked in the graph (Del), a random weighting of edges). A plotting field, where we draw a graph, is situated below the toolbar. In addition, we can save the graph or reload it

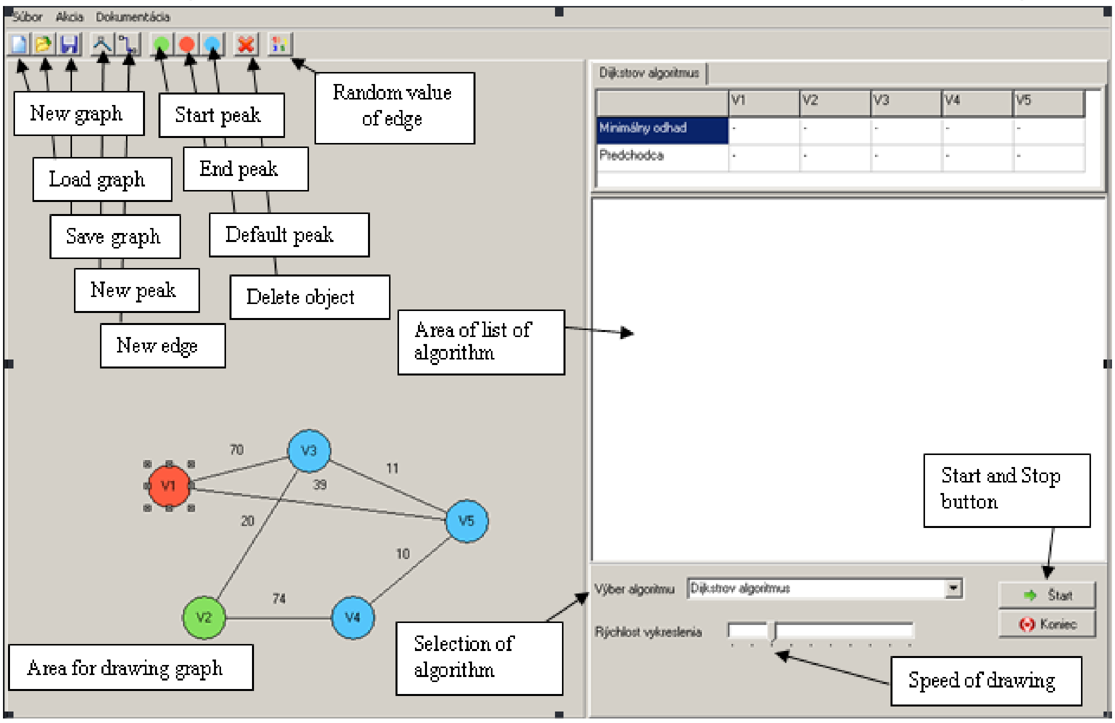

Fig. 7 Visualization graph in SimpleGraph

We can insert an arbitrary number of vertices into the graph. We link these vertices with dges by using the button "insert edges in a graph." We have an option of random weighting of the edges in the graph or we can weight them manually by clicking on the respective edge and entering its weight. We can see a process of the calculation on the figure (figure 8).

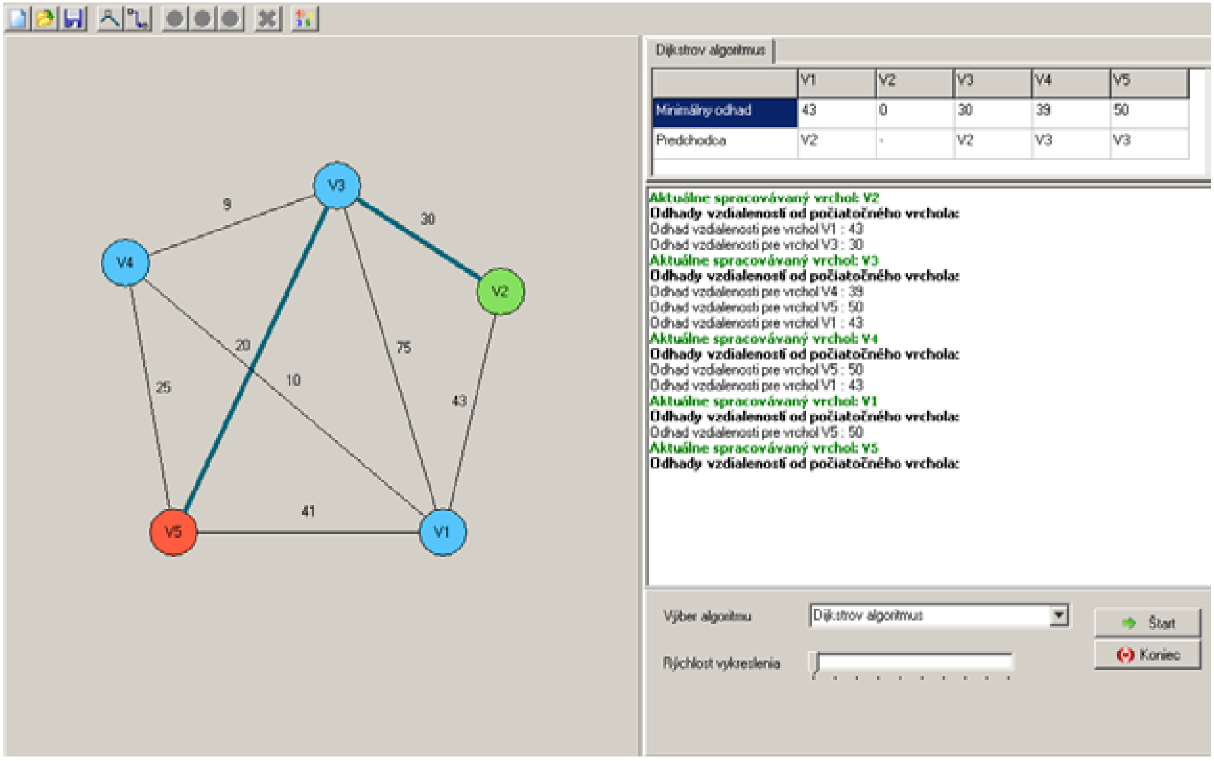

Fig. 8 *Visualization of Djikstric algorithm*

4 Conclusion

In this contribution we focused on the visualization in the teaching of the graph theory. A goal of application of these new approaches is to help a teaching process of informatics to become more demonstrative and appropriate with regard to the current trends in computer science and in the development of information technology. (VARGOVÁ, M., 2012) Taught topics, represented by the visualization, can be displayed in much more attractive and interesting way, which motivates students to learn. The appropriate visualization helps students to understand the presented problem better and more quickly. It makes it easier to understand causes and consequences of the changes. Interactive environment provides an immediate feedback and thus keeps student’s attention. The additional advantage is an option to move backwards in the process to the older solutions as well as to examine the accuracy of a solution. By using the tools for visualization and modeling we are able to enhance students’ ability to learn in a visual way along with their creative and logical thinking. The students can construct an algorithm of a given problem and to put down problem’s solution using the symbolic language.

References

DAUBNEROVÁ, Z. (2011). Visualization of the selected algorithms of graph theory in Imagine. Thesis. Ružomberok: Catholic university.

KRIŠŠ Š. (2013) Visualisation of the operations with matrix Bachelor‘s thesis. Ružomberok: Catholic university, 2013

LEHOTSKÝ, M., KRAKOVSKÝ, R., POVAŽAN, J.: Algoritmizácia a programovanie v jazyku PASCAL. Ružomberok : Katolícka univerzita v Ružomberku, 2006. 150 s. ISBN 80-8084-123-3.

PRÍLEPKOVÁ, K.(2012). Visualization selected algorithms for graph theory. Thesis. Ružomberok: Catholic university.

VARGOVÁ, M. (2012): Multicultural education and its application in pre-primary education. Ružomberok : Verbum - publishers Catholic university v Ružomberku, 2012. 232 s. ISBN 978-80-8084-916-0

LOTTO IN MATHEMATICAL TASKS OR MATHEMATICS WITHOUT COMPUTING

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Abstract:
This work shows some peculiar reasoning connected to mathematical problems inspired by the hazardous random game of LOTTO. It is about mathematics without computing.
Keywords: hazardous random game, random variable and its distribution, probability.
MESC: K10, K20

1 Introduction

Mathematics is usually associated with computing. Teaching of mathematics is targeted to development of computing skills. The capacity and ability to organize the phase of computing are easy in the area of controlling and evaluating students. But computing is only one of many mathematical activities which are to be taken into consideration while talking about education of and through mathematics.

In [1] the author states, beside computing and deduction, such mathematical activities as schematization and mathematization, discovering and justifying symmetries and analogies, concluding through symmetries and analogies, generalizing, defining, coding and decoding, algorithmizing.

The game of LOTTO, popular in Europe, is a source of many interesting mathematical problems and tasks. The following work concerns special ways of solving problems connected to hazardous games on the ground of mathematics but without calculations. What we are talking about here is *mathematics without computing*.

The notation of $U_{1\text{®}\,s}$ means an urn with s balls numbered from *1* to s. The notation of U_{b*c} means an urn with b white balls and c red balls.

2 LOTTO and the distribution of hits of a player who bought one coupon

The urn $U_{1\text{®}\,49}$ is a requisite in the LOTTO game. They draw six balls simultaneously or six times one ball without throwing it back in. The numbers on the drawn balls mean the numbers drawn.

Let us assume that they draw simultaneously six balls out of the $U_{1\text{®}\,49}$ urn. That is the d_6^{49} experiment. Before it takes place the player bets on the outcome by crossing out six numbers on the coupon he/she bought (pic. 2). If the number drawn is the same as the number the player crossed out, we call it *a hit*. The win of the player depends on the number of his hits. Having six hits the player wins the first class prize ([?], p. 248 – 249). So the player wins the first class prize if the d_6^{49}

drawing ends with exactly the same numbers he bet on with his coupon. We call it the $LOTTO_6^{49}$ game.

Pic. 1. A coupon for the LAJKONIK *game, Cracow, 1950s*

Let us assume that a player crossed out six numbers on his coupon, but the d_6^{49} game has not taken place yet. In this situation the number of the player's hits is a random variable which takes the values of 0, 1, 2, 3, 4, 5, 6. We need to notice that time is very important here.

One of the peculiarities of the $LOTTO_6^{49}$ game is the fact that the distribution of the number of the player's hits does not depend on which numbers he chose to cross out on his coupon. During classes with students we can prove that with complicated calculations (see [?], p. 251 – 252).

1	8	15	22	29	36	43
2	9	16	23	30	37	44
3	10	17	24	31	38	45
4	11	18	25	32	39	46
5	12	19	26	33	40	47
6	13	20	27	34	41	48
7	14	21	28	35	42	49

Pic. 2. The Lotto coupon

3 The $LOTTO_6^{49}$ player's situation mathematization and symmetries

Let us assume that the G_1 player crossed out the numbers: 2, 13, 17, 18, 27 and 44, and the G_2 player crossed out the numbers 1, 2, 3, 4, 5 and 6. It seems that the G_2 player's decision is not reasonable, as his chance to win the first class prize is smaller than the G_1 player's. We will prove this intuitional deduction incorrect.

Let us assume that the player crossed six numbers out on his coupon, but the drawing (the d_6^{49} experiment) has not started yet. At this moment the balls with chosen numbers are special for the player. We can say that they become red in the $U_{1® 49}$ urn, while the other balls remain white. The $U_{1® 49}$ urn now becomes the U_{43*6} urn. For the player (no matter which numbers he chose to cross out on his coupon) the six number $LOTTO_6^{49}$ drawing becomes a six ball drawing from the U_{43*6} urn, and the number of hits is the number of red balls among the ones drawn out.

The mathematization process described above allows us (with no formal calculations) to justify the fact that after crossing the numbers out but before the actual experiment the number of the player's hits is a random variable and its distribution does not depend on the actual numbers the player has chosen. In the $LOTTO_6^{49}$ game there is no rational strategy of winning depending on the numbers crossed out. So the G_2 player, who chose to cross out 1, 2, 3, 4, 5, 6 and the G_1 player who chose to cross out 13, 15, 22, 23, 39, 44 have exactly the same chance of having 6 hits, the same chance of having 5 hits etc.

4 Rationalization of procedures in the $LOTTO_6^{49}$ game

A lot of paper is wasted every year to produce coupons for hazardous number games. The view on the $LOTTO_6^{49}$ player's situation shown above proves that the procedures of the game could be substantially improved. Instead of crossing out numbers on a paper coupon the player could buy himself a chance of drawing six balls out of the U_{43*6} urn. The number of red balls drawn by him could be interpreted as the number of hits in one betting on six numbers out of 49 (that is six numbers crossed out on a coupon).

5 Two versions of a LOTTO game – analogies discovered without calculations

In a $LOTTO_6^{49}$ game the player bets on the outcome of a single drawing of six numbers out of 49 (the d_6^{49} experiment). Let us assume that there is another version of the game, in which the player bets on the outcome of a drawing of 43 numbers out of 49 (the d_{43}^{49} experiment). This is the $LOTTO_{43}^{49}$ game. In both games you bet by crossing out chosen numbers on a coupon from pic. 2. You get the first class prize if you cross all the numbers out correctly (you get six hits). It seems that the probability to get the first class prize in the $LOTTO_{43}^{49}$ game is much smaller than in the $LOTTO_6^{49}$ game.

Let us assume that a G_1 player of the $LOTTO_6^{49}$ game crossed out six numbers on his coupon and a G_2 player of the $LOTTO_{43}^{49}$ game crossed our 43 numbers on his one. Let us estimate the probability of getting the first class prize for both players. The G_1 player, who bet on the outcome of the d_6^{49} experiment, gets the first class prize if he has six hits. The G_2 player, who bet on the outcome of the d_{43}^{49} experiment, gets the first class prize if he has 43 hits.

We need to notice that when we draw six balls out of the $U_{1® 49}$ urn, the remaining 43 balls are also ones drawn out of it.

For both versions of the game there may be one common draw. If we draw six balls out of the $U_{1\to 49}$ urn we get the result of the $LOTTO_6^{49}$ game. But we can treat the remaining 43 balls as drawn out and so we get the result of the $LOTTO_{43}^{49}$ game. So there are as many results of the drawing in the $LOTTO_6^{49}$ and the $LOTTO_{43}^{49}$ games (here we discover an equality $\binom{49}{6} = \binom{49}{43}$) and all the results of both drawings are equally possible.

Let X be the number of hits of the G_1 player of the $LOTTO_6^{49}$ game who crossed out six numbers on his coupon and – similarly let Y mean the number of hits of the G_2 player of the $LOTTO_{43}^{49}$ game who crossed out 43 number on his coupon. For both players the $U_{1\to 49}$ urn becomes the U_{43*6} urn, but for the G_2 player the balls with the same numbers as he crossed out are white and the remaining ones are red, and for the G_1 the balls with numbers that he crossed out are red and the remaining ones are white. For both of them it is a drawing from the U_{43*6} urn.

If we draw six balls out of the U_{43*6} urn and there are j red balls among them (j = 0, 1, 2, 3, 4, 5, 6), there will be 43 – (6 – j) white balls among the 43 balls remaining in the urn (and these are also drawn out of this urn).

The number of red balls among the six drawn out of the urn is at the same time the number of the G_1 player's hits. The number of white balls among the 43 balls remaining in the urn is at the same time the number of the G_2 player's hits, so:

$$P(X = j) = P(Y = 43 - 6 + j) \text{ for } j = 0, 1, 2, 3, 4, 5, 6.$$

The above example shows that the probability of drawing six red balls out of the U_{43*6} urn in a single six-ball draw is exactly the same as the probability of drawing 43 white balls out of this urn in a single 43-ball draw.

The probability of getting the first class prize in the $LOTTO_6^{49}$ game is exactly the same as the probability of getting the first class prize in the $LOTTO_{43}^{49}$ game.

6 Conclusion

This work presents some examples of mathematical argumentations without computing. The bases for these argumentations are some stochastic analogies and symmetries.

References

Z. KRYGOWSKA, Elementy aktywności matematycznej, które powinny odgrywać znaczącą role w matematyce dla wszystkich, Dydaktyka Matematyki 6, 1986.

A. PŁOCKI, Prawdopodobieństwo wokół nas, Wydawnictwo dla Szkoły, Bielsko-Biała, 2011 (4 th edition).

A. PŁOCKI, Toto Lotek na lekcji matematyki, Nauczyciele i Matematyki + Technologia Informacyjna 67, 2008, p. 17 – 23.

A. PŁOCKI, P. TLUSTÝ, Kombinatoryka wokół nas, Wydawnictwo Naukowe NOVUM, PŁOCK, 2011.

A. PŁOCKI, P. TLUSTÝ, Pravděpodobnost a statistika prop začátečníky a mírně pokročilé, Prometheus, Praha, 2007

THE USE OF CRYPTOGRAMS WITH ARITHMETIC OPERATIONS IN SCHOOL TEACHING

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Abstract:
The text presents a method of solving cryptograms with arithmetic operations. This method is based on systems of equations. In the case of cryptograms with adding and subtracting operations we have to solve systems of linear equations while in the case of cryptograms with multiplying and dividing operations we have to solve systems of non-linear equations. The method of deciphering cryptograms will be illustrated by examples.

Key words: cryptograms, systems of linear equations, systems of non-linear equations.
MESC: F10, F90

1 Introduction

Cryptograms present an interesting kind of a mental pastime. Most frequently, they are solved by consecutive trials (a "guessing" method). Still, there is a systematic method of solving a large class of cryptograms by using systems of equations. In the case of adding and subtracting, deciphering cryptograms boils down to solving systems of linear equations while in the case of multiplying and dividing – to solving systems of non-linear equations. Solving the former, it is not necessary to present a theory of these, but it is worth presenting cases of determined equations (having a one unique solution) and underdetermined systems (having infinitely many solutions). In every case, we will assume that numbers used in cryptograms will not start with the digit 0. The symbols are to be replaced by digits so as to obtain a valid result of given operations. The additional assumption is that numbers hidden in symbols are written in the decimal system.

2 Examples of applications of the method

The method of solving cryptograms which involves arithmetic operations will be illustrated by examples.

Example no. 1. Decipher the following cryptogram knowing that for the numbers hidden above the line the sum of the units is greater than 10 and less than 20, and the sum of the tens is greater than 100 and less than 200.

$$\begin{array}{r} aab \\ acc \\ (+)cdb \\ \hline bcd \end{array}$$

Solution. The cryptogram may be presented as:

$$\begin{array}{r} 100a+10a+b \\ 100a+10c+c \\ (+)100c+10d+b \\ \hline 100b+10c+d \end{array}$$

Summing up the units, the tens and the hundreds under the conditions given, we will obtain:

$$\begin{cases} 2b + c = d + 10 \\ 10(a + c + d) + 10 = 10c + 100, \\ 100(2a + c) + 100 = 100b \end{cases} \quad \text{i.e.} \quad \begin{cases} 2b + c = d + 10 \\ a = -d + 9 \\ 2a - b + c = -1 \end{cases}.$$

Taking d as a parameter, we may present the solution of the latest system of linear equations as follows:

$$\begin{cases} a = 9 - d \\ b = \frac{1}{3}(29 - d) \\ c = \frac{1}{3}(5d - 28) \end{cases}.$$

For $d = 0, 1, 3, 4, 6, 7, 9,$ we do not get any natural number solution. For $d = 2, 5, 8$ we obtain respectively: $a = 7, b = 9, c = -6$; $a = 4, b = 8, c = -1$; $a = 1, b = 7, c = 4$. Only for $d = 8$ we obtain a solution in the set of natural numbers. So, this cryptogram has the form:

$$\begin{array}{r} 117 \\ 144 \\ (+)487 \\ \hline 748 \end{array}.$$

<u>Example no. 2</u>. Decipher the following cryptogram knowing that for the numbers hidden above the line the sum of the units is less than 10, and the sum of the tens is greater than 100 and less than 200.

$$\begin{array}{r} ABC \\ ACA \\ (+)CBD \\ \hline BDB \end{array}.$$

Solution. The cryptogram may be presented as:

$$\begin{array}{r} 100A+10B+C \\ 100A+10C+A \\ (+)100C+10B+D \\ \hline 100B+10D+B \end{array}.$$

Summing up the units, the tens and the hundreds under the conditions given, we will obtain:

$$\begin{cases} C + A + D = B \\ 10(2B + C) = 10D + 100 \\ 100(2A + C) + 100 = 100B \end{cases}, \quad \text{i.e.} \quad \begin{cases} A - B + C = -D \\ 2B + C = D + 10. \\ 2A - B + C = -1 \end{cases}$$

Taking D as a parameter we obtain infinitely many solutions of the latest system of linear equations.

$$A = D - 1, \ B = D + 3, \ C = -D + 4.$$

We find solutions in natural numbers less than 10 assuming that $A \neq 0$, $B \neq 0$ i $C \neq 0$, since the numbers hidden above the line do not start with the digit 0. From this, it easy follows that $D \in \{2, 3\}$. So, we have:

D	2	3
A	1	2
B	5	6
C	2	1

Therefore we obtained two solutions that fulfil the specified conditions:

$$A = 1, \ B = 5, \ C = 2, \ D = 2,$$
$$A = 2, \ B = 6, \ C = 1, \ D = 3.$$

The considered cryptogram may be rewritten as follows:

$$\begin{array}{r} 152 \\ 121 \\ (+)252 \\ \hline 525 \end{array} \quad \text{or} \quad \begin{array}{r} 261 \\ 212 \\ (+)163 \\ \hline 636 \end{array}.$$

<u>Example no. 3</u>. Find 8 solutions for each of the given cryptograms, knowing that in cryptogram (I) the sum of the units for the numbers hidden above the line is greater than or equal to 10 and less than 20, and the sum of the tens is less than 100. We assume that in cryptogram (II) the sum of the units for the numbers is less than 10 and the sum of the tens is less than 100.

$$\text{(I)}\quad \begin{array}{r} ABC \\ DAA \\ (+)DDC \\ \hline EEB \end{array} \qquad\qquad \text{(II)}\quad \begin{array}{r} xyx \\ zxz \\ (+)xxx \\ \hline uzu \end{array}.$$

Solution. Cryptogram (I). The following conditions must be fulfilled: $A > 0$, $D > 0$, $E > 0$, $B \geq 0$, $C \geq 0$. The given cryptogram can be represented as:

$$\begin{array}{r} 100A+10B+C \\ 100D+10A+A \\ (+)100D+10D+C \\ \hline 100E+10E+B \end{array}.$$

Summing up the units, the tens and the hundreds under the conditions given, we will obtain:

$$\begin{cases} 2C + A = B + 10 \\ 10(B + A + D) + 10 = 10E, \\ 100(A + 2D) = 100E \end{cases} \quad \text{i.e.} \quad \begin{cases} 2C + A = B + 10 \\ B + A + D + 1 = E. \\ A + 2D = E \end{cases}$$

Taking D and E as parameters, we have:

$$\begin{cases} A = E - 2D \\ B = D - 1 \\ C = \frac{1}{2}(3D - E + 9) \end{cases} \tag{1}$$

Thus, under the conditions given, the following inequalities are valid: $D \geq 1$ (since $B \geq 0$), $D < \frac{E}{2}$ (since $A > 0$). Moreover, since C is an integer number and $0 < E \leq 9$, so the number $3D - E + 9$ is a positive even number. Therefore, there exist the following cases:

	I	II	III	IV	V	VI	VII	VIII
E	9		8		7	6	5	4
D	2	4	1	3	2	1	2	1

The table below lists all the possible solutions of system (1):

	E	D	A	B	C
I	9	2	5	1	3
II	9	4	1	3	6
III	8	1	6	0	2
IV	8	3	2	2	5
V	7	2	3	1	4
VI	6	1	4	0	3
VII	5	2	1	1	5
VIII	4	1	2	0	4

The solutions of the given cryptogram are the following:

$$\begin{array}{r} 513 \\ 255 \\ (+)223 \\ \hline 991 \end{array} \qquad \begin{array}{r} 136 \\ 411 \\ (+)446 \\ \hline 993 \end{array} \qquad \begin{array}{r} 602 \\ 166 \\ (+)112 \\ \hline 880 \end{array} \qquad \begin{array}{r} 225 \\ 322 \\ (+)335 \\ \hline 882 \end{array}$$

$$\begin{array}{r} 314 \\ 233 \\ (+)224 \\ \hline 771 \end{array} \qquad \begin{array}{r} 403 \\ 144 \\ (+)113 \\ \hline 660 \end{array} \qquad \begin{array}{r} 115 \\ 211 \\ (+)225 \\ \hline 551 \end{array} \qquad \begin{array}{r} 204 \\ 122 \\ (+)114 \\ \hline 440 \end{array}$$

Cryptogram (II). From the previous assumptions it follows that > 0, $y \geq 0$, $z > 0$ i $u > 0$. The cryptogram may be represented as follows:

$$\frac{\begin{array}{r} 100x+10y+x \\ 100z+10x+z \\ (+)100x+10x+x \end{array}}{100u+10z+u}.$$

Summing up the units, the tens and the hundreds under the conditions given, we obtain:

$$\begin{cases} 2x + z = u \\ 10(2x + y) = 10z \\ 100(2x + z) = 100u \end{cases}, \qquad \text{i.e.} \qquad \begin{cases} 2x + z = u \\ 2x + y = z \end{cases}.$$

Taking z and u as parameters, we obtain:

$$\begin{cases} x = \frac{1}{2}(u - z) \\ y = 2z - u \end{cases} \tag{2}$$

Pairs of integers u, z, which satisfy the condition $2|(u - z)$ are the following:

$$(3,1); \quad (4,2); \quad (5,1); \quad (5,3); \quad (6,2); \quad (6,4); \quad (7,1); \quad (7,3);$$
$$(7,5); \quad (8,2); \quad (8,4); \quad (8,6); \quad (9,1); \quad (9,3); \quad (9,5); \quad (9,7).$$

Just some of these pairs satisfy the condition $2z - u \geq 0$. These pairs are:

$$(4,2); \quad (5,3); \quad (6,4); \quad (7,5); \quad (8,4); \quad (8,6); \quad (9,5); \quad (9,7).$$

Other pairs are invalid. So, we obtain the following solutions of system (2):

	I	II	III	IV	V	VI	VII	VIII
u	4	5	6	7	8	8	9	9
z	2	3	4	5	4	6	5	7
x	1	1	1	1	2	1	2	1
y	0	1	2	3	0	4	1	5

We present below all possible solutions of cryptogram (II):

I: $\frac{\begin{array}{r} 101 \\ 212 \\ (+)111 \end{array}}{424}$ II: $\frac{\begin{array}{r} 111 \\ 313 \\ (+)111 \end{array}}{535}$ III: $\frac{\begin{array}{r} 121 \\ 414 \\ (+)111 \end{array}}{646}$ IV: $\frac{\begin{array}{r} 131 \\ 515 \\ (+)111 \end{array}}{757}$

V: $\frac{\begin{array}{r} 202 \\ 424 \\ (+)222 \end{array}}{848}$ VI: $\frac{\begin{array}{r} 141 \\ 616 \\ (+)111 \end{array}}{868}$ VII: $\frac{\begin{array}{r} 212 \\ 525 \\ (+)222 \end{array}}{959}$ VIII: $\frac{\begin{array}{r} 151 \\ 717 \\ (+)111 \end{array}}{979}$

<u>Example no. 4</u>. Decipher the following cryptogram:

$$\frac{\begin{array}{r} xzz \\ yxx \\ (+)\ \ xy \end{array}}{ztt}.$$

Solution. The given cryptogram may be rewritten in the form:

$$\frac{\begin{array}{r} 100x + 10z + z \\ 100y + 10x + x \\ (+) \qquad 10x + y \end{array}}{100z + 10t + t}$$

Let $a \in \{0, 10, 20\}$ and $b \in \{0, 100, 200\}$. Because in this case there are no any restrictions (i.e. additional assumptions), then summing up the units, the tens and the hundreds under the conditions given, taking into account all cases, we will obtain:

$$\begin{cases} z + x + y = t + a \\ 10(z + 2x) + a = 10t + b \\ 100(x + y) + \mathrm{b} = 100z \end{cases}, \quad \text{i.e.} \quad \begin{cases} x + y + z = t + a \\ 2x + z = t + \frac{1}{10}(b - a) \\ x + y - z = -\frac{1}{100}b \end{cases}. \tag{3}$$

Taking t as a parameter, we obtain the following solutions of system (3):

$$\begin{cases} x = \dfrac{1}{4}t - \dfrac{3}{10}a + \dfrac{19}{400}b \\ \mathrm{y} = \dfrac{1}{4}t + \dfrac{8}{10}a - \dfrac{21}{400}b \\ \mathrm{z} = \dfrac{1}{2}t + \dfrac{1}{2}\left(a + \dfrac{1}{100}b\right) \end{cases}$$

If $a = 0$ and $b = 0$, then the solution will be of the form:

$$\begin{cases} x = \dfrac{t}{4} \\ \mathrm{y} = \dfrac{t}{4} \\ \mathrm{z} = \dfrac{t}{2} \end{cases}$$

Only for $t = 0$, $t = 4$ and $t = 8$ we obtain natural numbers from the interval $\langle 0, 9\rangle$. The case if $t = 0$ is omitted, because numbers do not start with the digit 0. There exist the following solutions: $t = 4,\ x = 1,\ y = 1,\ z = 2$ or $t = 8,\ x = 2,\ y = 2,\ z = 4$.
The cryptogram after deciphering has the form:

$$\begin{array}{r} 122 \\ 111 \\ \underline{(+)\ \ 11} \\ 244 \end{array} \quad \text{or} \quad \begin{array}{r} 244 \\ 222 \\ \underline{(+)\ \ 22} \\ 488 \end{array}.$$

If $a = 10$ and $b = 100$, then the solutions of system (3) are of the form:

$$\begin{cases} x = \frac{t}{4} - 3 + \frac{19}{4} \\ y = \frac{t}{4} + 8 - \frac{21}{4} \\ z = \frac{t}{2} + 5 + \frac{1}{2} \end{cases}.$$

Only for $t = 1$ and $t = 5$ we obtain natural numbers from the interval $\langle 0, 9\rangle$. Therefore, we have: $t = 1,\ x = 2,\ y = 3,\ z = 6$ or $t = 5,\ x = 3,\ y = 4,\ z = 8$. In this case there exist two cryptograms:

$$\begin{array}{r} 266 \\ 322 \\ \underline{(+)\ \ 23} \\ 611 \end{array} \quad \text{or} \quad \begin{array}{r} 388 \\ 433 \\ \underline{(+)\ \ 34} \\ 855 \end{array}.$$

In the remaining cases of a and b there are no natural numbers from the interval $\langle 0, 9\rangle$. Thus, given cryptogram has only 4 solutions.

Example no. 5. In the given cryptogram the sum of the units in the minuend is less than the sum of the units in the subtrahend.

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Solution. Let § $= x$, © $= y$, ¨ $= z$. Now, the given cryptogram may be presented as follows:

$$\begin{array}{r} 100x+10y+z \\ \underline{(-)\quad 10y+y} \\ 100y+10x+x \end{array}.$$

Under the assumption that $z < y$, the cryptogram may be rewritten in the form:

$$\begin{cases} (z + 10) - y = x \\ 10(y - 1) + 100 - 10\mathrm{y} = 10x, \\ 100(x - 1) = 100y \end{cases} \quad \text{i.e.} \quad \begin{cases} x + y - z = 10 \\ x = 9 \\ x - y = 1 \end{cases}.$$

The last system of equations has the following solution: $x = 9,\ y = 8,\ z = 7$. So, the deciphered cryptogram has the form:

$$\begin{array}{r} 987 \\ \underline{(-)\ \ 88} \\ 899 \end{array}.$$

<u>Example no. 6.</u> Decipher the following cryptogram: $ac \cdot cb = cab$. The additional assumption is that the products $a \cdot b,\ \ a \cdot c,\ \ b \cdot c$ are less than 10.

Solution. The given cryptogram can be expressed as the following equality:

$$(10a + c) \cdot (10c + b) = 100c + 10a + b,$$

i.e.

$$100(a \cdot c - c) + 10(a \cdot b + c^2 - a) + b \cdot c - b \equiv 0.$$

Since each of the products is less than 10, then:

$$\begin{cases} a \cdot c - c = 0 \\ a \cdot b + c^2 - a = 0, \\ b \cdot c - b = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} c(a-1) = 0 \\ a(b-1) + c^2 = 0. \\ b(c-1) = 0 \end{cases}$$

So we obtain the system of non-linear equations. From the first equation of the last system it follows that $c = 0$ or $a = 1$. Since $c \neq 0$ (numbers do not start with the digit 0), thus $a = 1$. Therefore we get the following system:

$$\begin{cases} b - 1 + c^2 = 0 \\ b(c-1) = 0 \end{cases}.$$

This system has the two solutions: $\begin{cases} b = 0 \\ c = 1 \end{cases}$ and $\begin{cases} b = 0 \\ c = -1 \end{cases}$. Only the solution $\begin{cases} b = 0 \\ c = 1 \end{cases}$ fulfils the condition $c > 0$. The given cryptogram has the form $11 \cdot 10 = 110$.

<u>Example no. 7</u>. Decipher the following cryptogram $abc \cdot ab = dcbc$ under the condition $c > 2a \cdot b$.

Solution. The cryptogram can be expressed as the equality of the form:

$$(100a + 10b + c) \cdot (10a + b) = 1000d + 100c + 10b + c,$$

i.e.

$$1000(a^2 - d) + 100(2a \cdot b - c) + 10(b^2 + a \cdot c - b) + b \cdot c - c \equiv 0.$$

Under the previous assumption of the form $2a \cdot b - c < 0$, the last identity can be rewritten as follows:

$$1000(a^2 - d) + 10(20a \cdot b - 10c) + 10(b^2 + a \cdot c - b) + b \cdot c - c \equiv 0,$$

from where we get the system:

$$\begin{cases} a^2 - d = 0 \\ 20a \cdot b - 10c + b^2 + a \cdot c - b = 0 \\ b \cdot c - c = 0 \end{cases} \qquad (4)$$

From the last equation we obtain the results: $c = 0$ or $b = 1$. So we need to consider two cases.

i. $c = 0$.

The first two equations of system (4) have the forms:

$$\begin{cases} d = a^2 \\ b(20a + b - 1) = 0 \end{cases}.$$

From the second equality we obtain: $b = 0$ or $20a + b - 1 = 0$. The case $20a + b - 1 = 0$ is impossible, since for $a > 0$ we will get $b < 0$ – contradiction (b is a digit). So, the conditions $c = 0,\ b = 0$ and $a = \sqrt{d}$, imply that $d = 1$ or $d = 4$ or $d = 9$. In this case we have: $a = 1$ or $a = 2$ or $a = 3$. Under these results the possible forms of the cryptogram are: $100 \cdot 10 = 1000$; $200 \cdot 20 = 4000$, $300 \cdot 30 = 9000$.

ii. $b = 1$.

In this case the first two equations of system (4) have the following form:

$$\begin{cases} d = a^2 \\ 20a - 10c + a \cdot c = 0 \end{cases}, \quad \text{i.e.} \quad \begin{cases} d = a^2 \\ c = \frac{20a}{10-a} \end{cases}.$$

Under the condition $0 < d < 10$ it follows that $a = 1, a = 2, a = 3$. Then $d = 1$, $d = 4$, $d = 9$, $c = \frac{20}{9}$, $c = \frac{40}{8} = 5$, $c = \frac{60}{7}$. Only $c = 5$ fulfills the previous conditions. In the case (II) system (4) has the solution $a = 2, b = 1, c = 5$. Thus the cryptogram has the form $215 \cdot 21 = 4515$.

<u>Example no. 8.</u> Decipher the following cryptogram $\frac{ab}{ca} = c$.

Solution. This cryptogram can be replaced with the cryptogram $ab = ca \cdot c$. Then $10a + b = (10c + a) \cdot c$, i.e. $10a + b = 10c^2 + a \cdot c$. From this we obtain the system: $\begin{cases} a = c^2 \\ b = a \cdot c \end{cases}$, i.e. $\begin{cases} a = c^2 \\ b = c^3 \end{cases}$. If $c \geq 3$, then $b > 10$. There are two cases for $c > 0$: $c = 1$ and $c = 2$. For $c = 1$ we obtain $a = 1, b = 1$; while $c = 2$ we have $a = 4$, $b = 8$. The given cryptogram has the solutions: $\frac{11}{11} = 1$ and $\frac{48}{24} = 2$.

3 Conclusion

As can be seen, this method is effective. At school, one presents in details the methods of solving systems of two linear equations with two variables and also presents determined, underdetermined and overdetermined systems. Widening the methods to cover also the systems of three equations with three variables causes no problem to the pupils. School teaching concerning solving these might be made more attractive by solving cryptograms. Often, systems of linear equations appear at the occasion of investigating the linear function. The problems of deciphering cryptograms may be discussed at the lessons themselves or at additional school meetings (special interest groups) and after school events (various contests and competitions).

Exercises.

Decipher cryptograms under given conditions.

1. $\frac{\begin{array}{r} ABB \\ CBC \\ (+)DBC \end{array}}{ECE}$, knowing that for the numbers hidden above the line the sum of the units is less than or equal to 10, and the sum of the tens is less than or equal to100.

2. $\frac{\begin{array}{r} ab \\ bca \\ (+)\,bab \end{array}}{adc}$, knowing that for the numbers hidden above the line the sum of the units is greater than 10 and less than 20, and the sum of the tens is greater than 200.

3. $\frac{\begin{array}{r} xyz \\ yxz \\ (+)zyy \end{array}}{uzv}$, knowing that for the numbers hidden above the line the sum of the units is less than or equal to 10, and the sum of the tens is greater than 100 and less than 200.

4. $\frac{\begin{array}{r} abb \\ (-)\ cac \end{array}}{cca}$, where $b > a$ and $b > c$.

5. $\begin{array}{r} pqr \\ (-)\ rps \\ \hline qpq \end{array}$, where $r > s$ and $q < p$.

6. $aaa \cdot aaa = abcba$.

7. $ab \cdot bb = acb$.

Answers:

1. $A = 2,\ B = 1, C = 3, D = 2, E = 7$.
2. $a = 8,\ b = 3,\ c = 4,\ d = 1$.
3. $x = 1,\ y = 5,\ z = 1, u = 8,\ v = 7$ or $x = 3,\ y = 4,\ z = 1,\ u = 9,\ v = 6$.
4. $a = 2,\ b = 3,\ c = 1$ or $a = 4,\ b = 6,\ c = 2$ or $a = 6,\ b = 9,\ c = 3$.
5. $p = 6,\ q = 2,\ r = 3,\ s = 1$.
6. $a = 1,\ b = 2,\ c = 3$.
7. $a \in \{1, 2, 3, 4, 5, 6, 7, 8\},\ b = 1,\ c = a + 1$ (multiplying a double digit number ending in 1 by 11).

References

GLEIGHGEWICHT, B. Algebra. PWN, Warszawa 1983. ISBN 83-01-03903-5.

Robert Sochacki

TEACHING PROBABILITY VIA PROBLEM SOLVING

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Abstract:
Problem solving is a mathematical process in which we construct our knowledge through our experience. It is an interesting and enjoyable way to learn mathematics. Most mathematical problems have more than one solution. Each of them can be approached in a variety of ways. Students can find one approach which helps them to solve a particular problem. Having seen various students' solution approaches in time, you should be able to develop and extend a range of strategies that the students have at their disposal. We will demonstrate this approach on some problems from the theory of probability.
Key words: Probability, problem solving, symmetry.
MESC: K10, K20, D40

1 Introduction

According to the results of TIMSS and PISA international researches, Czech pupils' motivation for education has been decreasing and it is apparent mainly in the relation to mathematics. Between 1995 and 2007 years, the Czech Republic noticed the second biggest decrease of the results within the countries of Central and Eastern Europe which participated in TIMSS testing of the eighth grade classes. According to Czech teachers of maths, teaching is negatively affected by several factors, reduced hour subsidies of maths while maintaining the amount of the taught subject-matter is one of them. The lack of time, permanent hurry and the effort to master everything which is prescribed have a devastating effect on the very subject-matter and meaning of teaching maths.

Students are increasingly presented to understand maths as a set of algorithms and proceedings that help to solve "standard" school problems.

Generally, students are first introduced to a theory and a process solution of typical tasks, later the rate of understanding and mastering of subject-matter is validated on analogous examples. Usually, there is not much space both for students' independent thinking and creativity, necessary for solving atypical or more difficult problems, or for searching for other possible solutions. It leads to the situation that students try to apply the procedures they have learned and have only very little experience with finding their own approaches or other solution possibilities. Such style of teaching mathematics, in our opinion, significantly weakens its main role and purpose in the educational process. The main objective of teaching mathematics is not learning of any set of algorithms, calculation procedures and methods but developing intellect, creativity and the ability to deal with the problem.

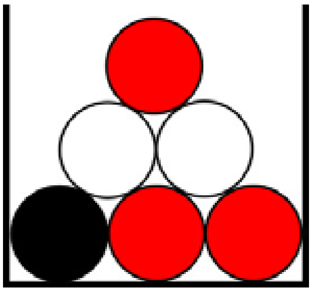
Should we intend to change this negative trend in the education of maths, first of all it is necessary to begin with teachers of maths and students of pedagogical faculties. It is clear that some parts cannot do without algorithms and formulas, however, there are some areas of mathematics in which only a small modification of the task does not allowed to "apply" formulas or some routine procedure. It especially holds for geometry, combinatorics probability, etc. We believe that it is also one of the reasons why these areas of mathematics are less popular among students than some others. Development of students' independent thinking, problem solving and searching for new methods of solution have been involved in a seminar on the theory probability. We are going to present two examples that we have solved together with students in a classical way. Then we have encouraged the students whether (by now with the knowledge of the result) they could suggest a different solution procedure, i.e. we aimed to make them to think about the given problem once again.

2 Assignment of problems

Problem 1:

There are three red, two white and one black balls in the box on the picture number 1. For three times we draw one ball, after each draw we do not return the ball back into the box. What, then, is the probability that the drawn ball will be

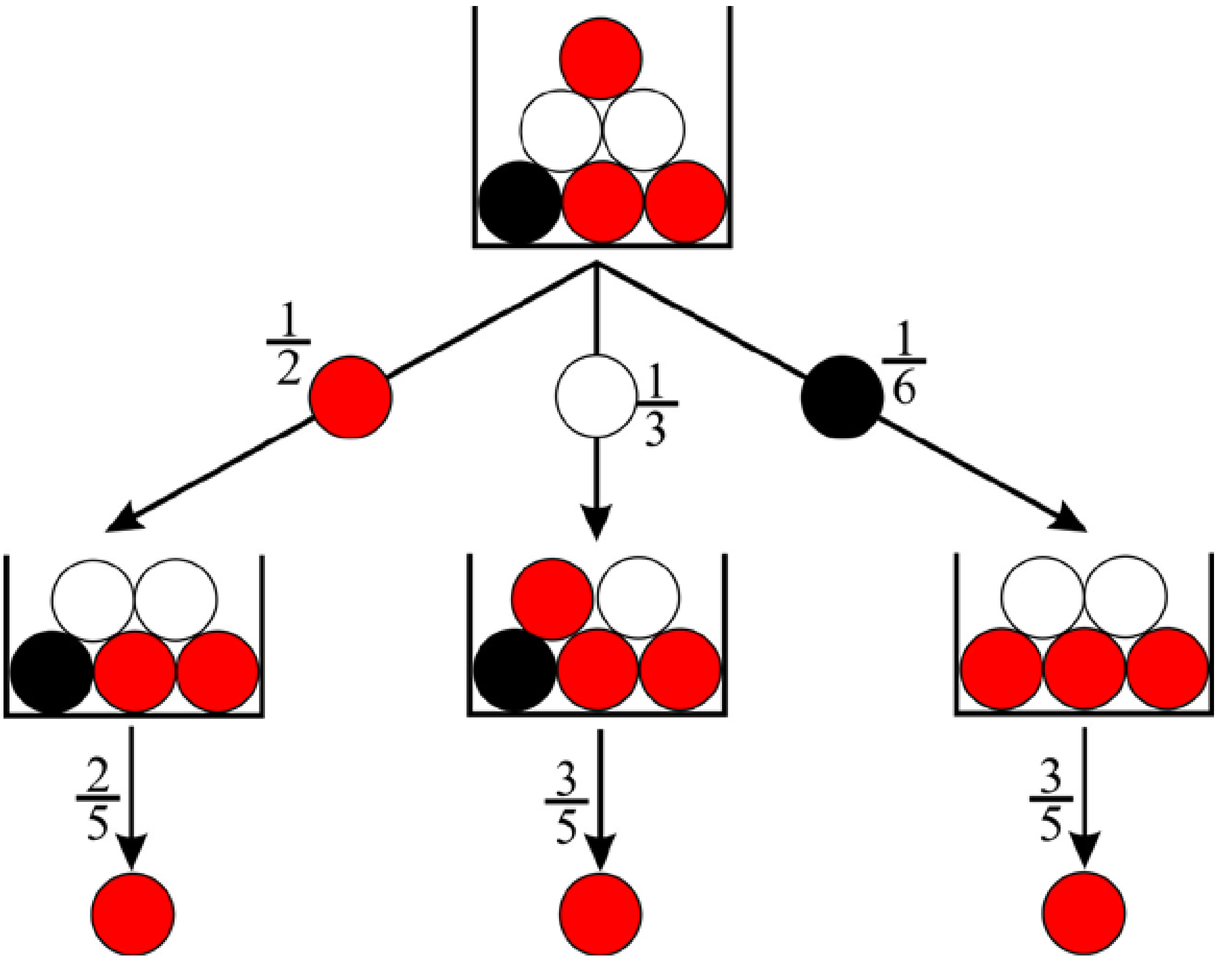
a) red in the second draw,
b) black in the third draw?



Pic.1

Solution:

a) It is clear that the searched probability depends on which ball (more accurately which ball colour) was drawn in the thirst draw. It is the conditional probability and to be able to solve it we can use either the total probability theorem (see [2]) or stochastic tree (Picture number 2).



Pic. 2

The stochastic tree shows that the searched probability equals to

P (the second ball is red) = $\frac{1}{2} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{3}{5} + \frac{1}{6} \cdot \frac{3}{5} = \frac{1}{2}$

The situation a) has been successfully solved but we stayed with it for a moment and allowed the students to discuss the problem solution, search for general principles and mission or find an entirely new solution.

Discussion:

1. Having seen the picture number 2 and having made the consecutive calculation, some students realized that the ball paint is not completely essential in the meaning that only red and “not-red” balls are important. Thus, the result does not depend on the specific color of "not-red" balls. If we realize this fact then we actually have the box with three red and three gray balls.

Having realized the symmetry, it is clear that the searched probability is equal to $\frac{1}{2}$ and we do not need either the picture number 2 or consecutive calculation.

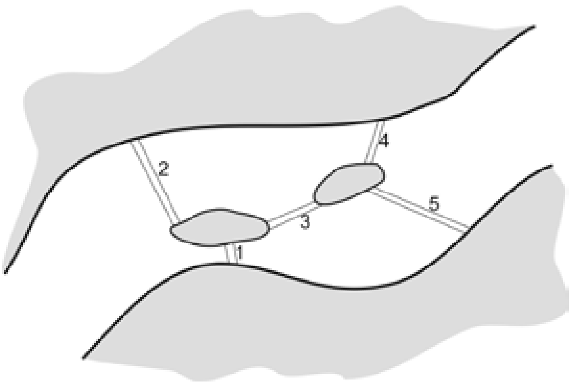
2. If we generalize the previous consideration it is clear that there is no reason to believe that some of the balls would be more likely pulled in the second draw than another ball. Thus, drawing

of any of the six balls is equally probable, and as we have just 3 red ones the searched probability equals $\frac{3}{6}$, i.e. $\frac{1}{2}$.

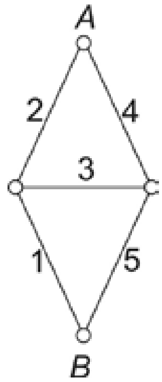
b) Obviously, it is possible to apply the above mentioned consideration from 2 for the third draw, i.e. the probability that we will pull the black ball in the third draw equals $\frac{1}{6}$. It is useful to realize how complicated it would be to get this result while using the formulas for the conditional probability.
Note that the above mentioned considerations are of general validity and can be used for the entire number of similar problems.

Problem 2:

There is a city plan with a river and two islands in the picture number 3. There are five bridges built on the river. When flooded, the probability of each bridges' destruction equals to 1/2 (the destruction or preservation of one bridge does not affect the destruction or preservation of another one). What is the probability that we will be able to walk from one river bank to another one after the flood?



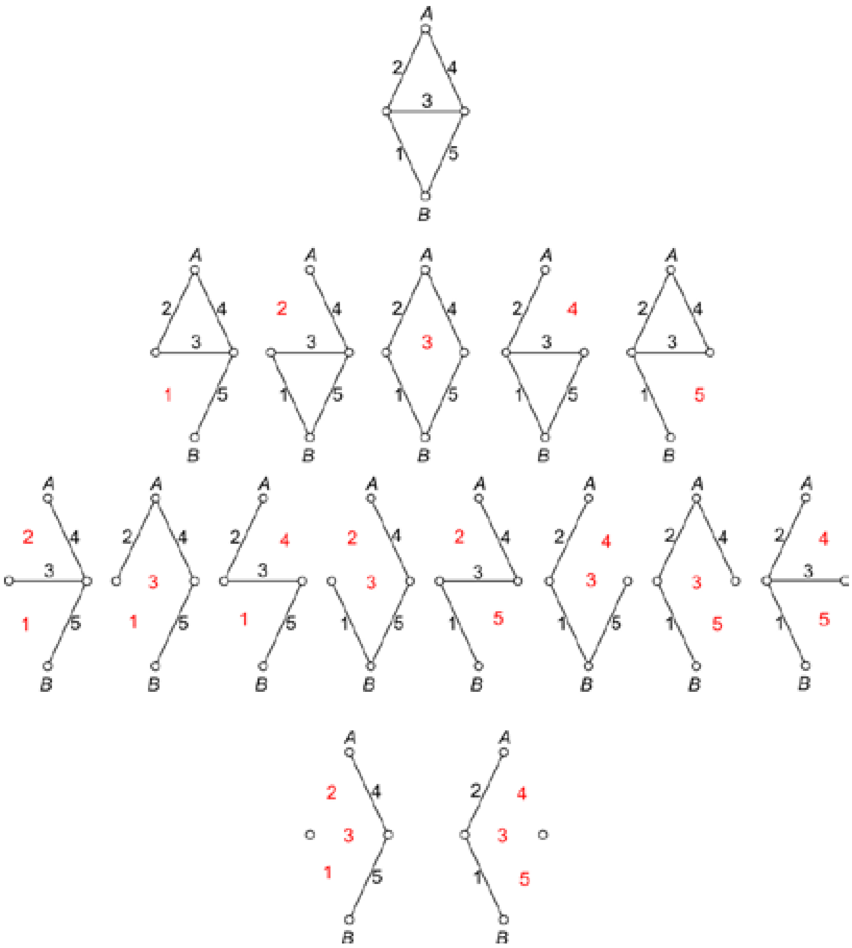
Pic. 3



Pic. 4

Solution:

There is a schematic situation illustrated as a graph in the picture number 4, the points *A* and *B* represent opposite sides of the river. Each of the 5 bridges can be destroyed or not, i.e. we have to distinguish 2^5, i.e. 32 different cases. Considering the fact that the bridges' destruction happens independently on each other and with the same probability $\frac{1}{2}$, each of the 32 possibilities can happen in the equal probability. Therefore we have a classic probability space and it remains to determine the number of cases when we can walk from the point *A* to the point *B*. All favourable possibilities are depicted in the picture number 5 where each line represents the favourable possibilities while destruction of none, one, two and three bridges. The destroyed bridge is marked in red.



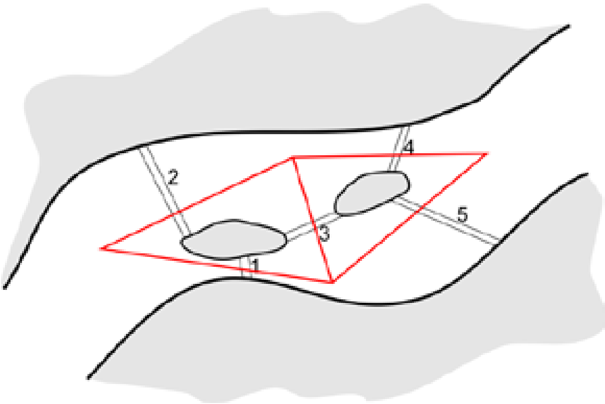
Pic. 5

As it is evident from the picture number 5 there is a total of 16 favourable possibilities, i.e. the probability that once flooded we will be able to cross the river equals $\frac{16}{32}$, i.e. $\frac{1}{2}$.

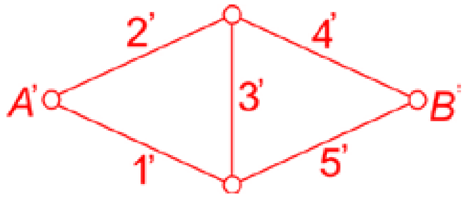
Discussion:

The question remains whether now when we know how the searched probability is big, it is possible to find another "more elegant" solution. The inability to move from one side of the town to the other one is for its residents certainly uncomfortable but look at this situation from a different angle. Imagine that a transport company has to carry an oversized load over the river but it fails to fit under each of the bridges. The situation of such company is exactly the opposite of the city residents as we try to find out the probability the load will be carried on the river.

The situation is schematically illustrated in the picture number 6. The transport company needs a clear passage from the point A' to the point B'. If we consider “the duality" between the transport company and the city residents then, after comparing the pictures number 4 and 7, it is clear that if the residents can cross the river, the transport company will not be able to carry the oversized load and vice versa. Thus, even without counting it is clear that the probability that once flooded we will be able to cross the river equals $\frac{1}{2}$.



Pic. 6



Pic. 7

References

A. ENGEL. STOCHASTICS. ERNST KLETT VERLAG, Stuttgart, 1987

A. PŁOCKI, P. Tlustý. Pravděpodobnost a statistika pro začátečníky a mírně pokročilé. Prometheus, Praha, 2007

DOMINANT FREQUENCY EXTRACTION

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Abstract

Time series are collected and studied extensively for the knowledge about the data source characteristics such as the trend or the spectral landscape. Some peaks in the spectral landscape correspond to dominant frequencies. The approach here is empirical: all time series are discrete and finite Contents: Introduction. 1 Examples of periodic phenomena. 2 Algorithms and libraries. 3 Time series analysis. 4 Dominant frequency in ladar data. Conclusion. References.

Keywords: Dominant Frequency, Dominating Frequency, Dominance Frequency, Peak Frequency, Spectral Peaks, Spectral Spikes.

MESC: D40, E10, E20

1 Introduction

While science tries to find the truth about the nature and the universe, the engineering approaches the properties of matter and phenomena which are necessary for human accomodation and survival, and which we can create and/or change. However, both science and engineering are simplifying our image of the world, sometimes into absurds. One way of simplification is to characterize a situation or a phenomenon by a single useful number, for example, a proportion, a volume, an average speed, an estimated probability, etc. Sometimes we find few numbers giving the useful characteristics. The observation of changes in the world leads to collections of data stamped by time; these data are called *time series*. The time interval between data is mostly constant, but this restriction can be worked out.

We are interested here in phenomena called periodicity, periodic events, cyclical components, oscillations, vibrations, rhythms, resonance, seasonal variations, etc. The books on vibrations and waves which combine mathematics and nice illustrations are [2, 4]. There are too many papers on detection of periodic motion in video images; let’s just mention these [3, 10, 11]. There are several weaker notions of periodic functions which we do not consider here. These are: almost periodic, pseudo periodic, quasi periodic, nearly periodic, and semi periodic. The extraction of the dominant frequency is just a small part of knowledge extraction from time series [6].

The temporal or dynamic changes in data can reflect a *harmonic motion*, which has a sinusoid in some distorted form. A sine wave can be a projection of circular motion plot against time. A *periodic motion* is all motion that repeats periodically. This includes the harmonic motion and pulses. While a pendulum of a wind-up clock performs a harmonic motion, the pendulum watched through a narrow window in front of the clock appears as pulses, which are periodic, but not harmonic. A *random motion* occurs in erratic manner, as a sampling to an unknown probability distribution, and may contain all frequencies in a particular band. The main feature of a random motion is that it is not repeated or repeatable, in another words, there are no similarities of ups and downs (mountains and valleys) in different time intervals. The non-periodic non-random motions are piecewise linear or piecewise curved.

Here we consider (discrete) time series of finite length resulting from a data collection - measurement and recording. There are many situations when the observed data show a periodic behavior due to one frequency, called the *dominant frequency*, which carries the maximum energy among all frequencies found in the spectrum. A similar notion is the *fundamental frequency*, which is the smallest frequency having a peak among all frequencies in a power spectrum. The fundamental frequency is found in a vibrating string or organ pipe, along with weaker frequencies being the *harmonics* (i.e., multiples) of the fundamental frequency (*see harmonic* in [14]). The presence of harmonics in the signal demonstrates itself in the periodicity of the power spectrum, which has a comb of spikes. The usual technique for the detection of the fundamental frequency and its harmonics is the *cepstrum*. There can be, however, other frequencies than harmonics present in the time series. They can be ordered by their energies, like it is done in electroencephalograms. We call them the *2nd dominant*, the *3rd dominant*, etc. When there are too many frequencies to characterize the periodicity of the data (time series), we can use the *mean frequency*. The *extraction of dominant frequency* means finding it, and sometimes removing it from the data as we remove linear trends. In that case, the plot of data with dominant frequency removed is called the *residual plot*. It is sometimes more appropriate to replace the word “extraction” by *determination* or *estimation*. Other notions used for dominant frequency are dominating frequency, dominance frequency, dominant spike, or peak frequency. The dominant frequency can be the *result of a resonance*, which can be desired (violins, vibraphones), or undesired (bridges, buildings), but it is always an important phenomenon.

Whether the dominant frequency is seen in plots of data or not, the best way to reveal it is via a spectrogram (from Fourier transform) or a scalogram (form a wavelet transform). The *spectral analysis* of data can reveal a hidden periodicity in data, so finding the dominant frequency is sometimes called *spectrum peak picking*. The Fourier transform maps the data into the complex number domain. We can look at its real part and imaginary part, the amplitudes of frequencies and the phases of frequencies. The squares of amplitudes are called the *power spectrum* or *periodogram*. It is the computers and programming languages which started in 1950’s the advanced data analysis a new science was born called *Digital Signal Processing* (DSP), and little later, Digital Image Processing.

2 Examples of periodic phenomena

- A pendulum or any mechanical clock are suitable for testing some algorithms or demonstration of principles.
- The heartbeat - the data is generated either by acoustic or electrical devices. The electrocardiogram (ECG) provides multiple time series. It is important for atrial fibrillation (see [14]) and ventricular fibrillation (see [14]) analyzed in cardiology.
- The breathing - the activity of lungs is recorded by monitoring the flow of air, the movements of chest, or electrical impulses.
- The electrical activities of brain are monitored by the electroencephalogram (EEG).
- The electrical activities of digestive system are monitored by electrogastrogram (EGG).
- The sleep laboratories have many different kinds of physiological devices and recorders, called polysomnograms, which include electroculograms (EOG) for recording of eye movements, and electromyograms (EMG) for the electrical activities of skeletal muscles.
- The speech analysis has many uses. The analysis of acoustic recordings helps to recognize elements of speech for speech to text translations, and for logopedics.
- The flapping of wings of birds and buzzing of insects. This is done with acoustic and/or video recording equipment.
- The wavy motions of fish, octopus, crabs, and other sea creatures are studied in marine biology.
- The elapsed-time photography helps to study slow-changing features in biology, geology, meteorology, etc.
- The sunspots (protuberances) are monitored with multi-resolution cameras on the orbits of the Sun. We try to predict their occurrence and intensity, because of their negative impact on electromagnetic devices on Earth.
- The vibrations of buildings, bridges, towers, constructions, machines, trucks, cars, etc. These are studied by vibrometry, which is both science and engineering domain.
- The seasonal components of time series collected from industrial, commercial, financial, communication, transport of people and goods, energy consumption, road traffic at crossings, and other activities of human society.
- The seismic activities of Earth and eruptions of volcanoes are not periodic when monitored in 24 hours/day, but they contain periodic segments of data, including infrasounds. The purpose of seismology and volcanology is to study the early signs of seismic activities in order to predict the earthquakes, and request evacuation of people in affected areas.
- The meteor showers, the ocean waves, the ocean streams, and atmospheric winds are important for weather tracking. The weather patterns are too complicated to be characterized by time series. The observation of sunny and cloudy days is not sufficient for prediction of weather tomorrow and few days ahead. Even if they are reoccurring, they are not periodic events. However, the weather science (meteorology) makes constant progress with amount and speed of data processed for more accurate and useful forecasts.
- The pattern recognition and texture characterization are branches of image processing, where the Fourier spectra are 2-dimensional, as well as other trans- forms and characteristics.

3 Algorithms and libraries

In the empirical approach we use the professional software libraries in suitable programming languages to analyze the time series, and in particular, to detect the dominant frequencies. The standard tools are: Fast Fourier Transform, Short- Time Fourier Transform, spectra, periodograms, and scalograms of various wavelets. These are included in many modern programming languages like C, C++, C#, IDL, Java and Python. Then there are special languages for statistical and digital signal computing, like S-plus (a commercial implementation of the S programming

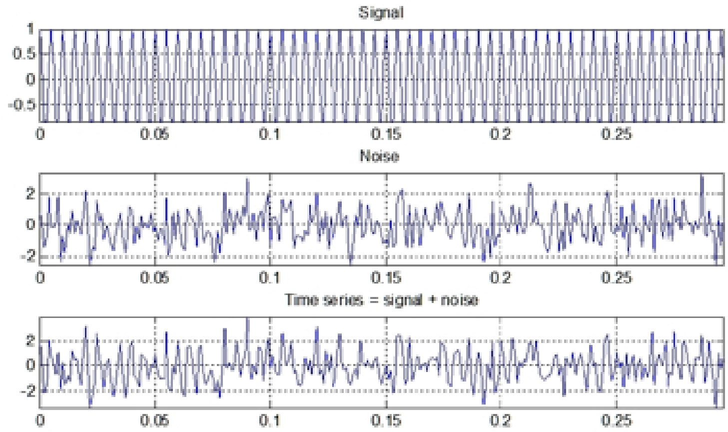

Fig. 1 The creation of time series with a dominant frequency

language) and R language (free). There are many commercial software packages available for statistical analysis with great graphics. The most popular in research, engineering and education is Matlab from MathWorks [8] with DSP, Image Processing, Wavelet, and other toolboxes. Moreover, there is constant flow of contributions from active users in all over the world. Good libraries and contributed codes are also available for Octave (free download), which is an alternative interpreter of Matlab m-files. For the time series occurring in economy there was developed the Berlin Procedure (Berlin Verfahren), which can perform seasonal adjustment to data, for example, daily, weekly, monthly or quarterly, and then perform an in-depth analysis. The latest version of the Berlin Procedure, BV4.1, is available for free download for non-commercial purposes. It turns out that market time series and not so random as they look in short time intervals. However, the prediction of time series is quite different problem than an extraction of the dominant frequency. Actually, the dominant frequencies in market data are call seasonal variations, such as monthly, quarterly and yearly. These are removed from the data, and then we have the residual time series, where we look for periodicities and other attributes.

When a time series is collected from a relatively unknown source, it may have various components and their proportions what makes the extraction of dominant frequency quite difficult. On the other hand, we can create an artificial discrete time series of finite length by taking a finite combination of sinusoids representing different frequencies, and add a reasonably small Gaussian white noise. Then the sinusoid with the largest amplitude (the absolute value of the coefficient) is the dominating frequency. A typical entry level example consist of one sinusoid plus a Gaussian white noise (Figure 1). The power spectrum of this time series has one pronounced peak clearly showing the dominant frequency (Figure 2). Here is the corresponding Matlab code:

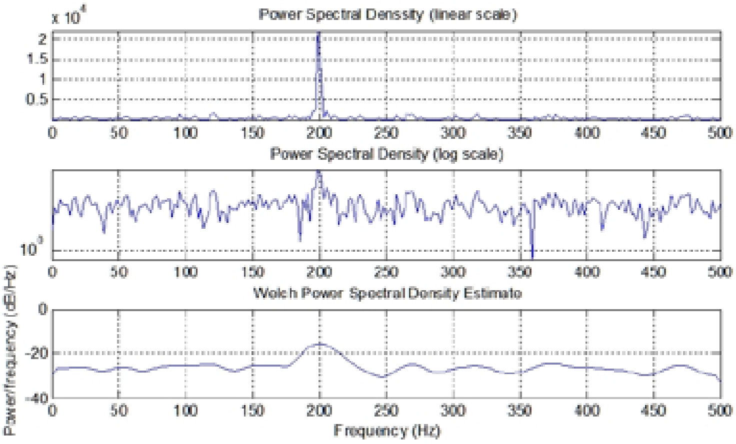

Fig. 2 The power spectra with the dominant frequency

```
function demo_dfe
Fs = 1000; % sampling frequency 1 kHz
t = 0 : 1/Fs : 0.296; % time scale
f = 200; % Hz, embedded dominant frequency
x = cos(2*pi*f*t) + randn(size(t)); % time series
plot (t,x), axis('tight'), grid('on'), title('Time series'), figure
nfft = 512; % next larger power of 2
y = fft(x,nfft); % Fast Fourier Transform
y = abs (y.^2); % raw power spectrum density
y = y(1:1+nfft/2); % half-spectrum
[v,k] = max(y); % find maximum
f_scale = (0:nfft/2)* Fs/nfft; % frequency scale
plot(f_scale, y),axis('tight'),grid('on'),title('Dominant Frequency')
f_est = f_scale(k); % dominant frequency estimate
fprintf('Dominant freq. : true %f Hz, estimated %f Hz\n', f, f_est)
fprintf('Frequency step (resolution) = %f Hz\n', f_scale(2))
```

The thoughts about artificial mixtures of sinusoids and a white noise are not without a merit. The Pisarenko Harmonic Decomposition (PHD) does precisely that (see [14]), even more, it can tell amplitudes and frequencies of dominant frequencies up to certain number, which should be always specified ahead and should be small relative to the length of data. The Multiple Signal Classification (MUSIC) is a generalization of the Pisarenko Harmonic Decomposition. MUSIC accepts the complex-valued time series which are supposed to be the sum of p complex exponentials and a complex Gaussian white noise. The algorithm returns p largest peaks. The MUSIC algorithm is implemented in Matlab's Signal Processing Toolbox as spectrum.music [7].

4 Time series analysis

The data (time series, signal) may come from another engineering team, or we might create them by running a possibly faithful simulation of the engineering process. Thus, in both cases, the data consist of finitely many samples, but they can be multi-dimensional, where the dimension is the number of features tracked. Let's focus on one dimensional case, where we study the temporal (dynamic) properties of the data source, and we focus on finding the dominating frequency. The task of dominant frequency extraction may look simple like this: find the peak in the power spectrum density of the signal to get the frequency, and then read the corresponding amplitude and phase from the Fourier transform of the signal. In cases where is a small number of fairly distant frequencies and a low value of the noise it is indeed so simple. However, in most important practical cases we need to discuss a number of issues.

4.1 The characteristics

n_S = (number of samples in the data) = (length of the data sequence)

T_S = (time interval of collecting the samples) = (the observation time)

r_S = (sampling rate) = n_S/T_S

f_N = (Nyquist frequency) = $r_S/2$

f_{max} = (maximal frequency computable with FFT-based spectrum analysis) = f_N

t_S = (time step) = (time between samples) = $T_S/n_S = 1/r_S$

p_f = (duration of one period of a frequency f) = $1/f$ seconds

np_f = (number of periods of a frequency f during time T_S) = $T_S/p_f = T_S f$

ns_f = (number of samples in one period of frequency f) = n_S/np_f

f_{res} = (frequency resolution in Fourier spectrum analysis) = $1/T_S$ Hz

f_{min} = (minimal frequency computable via Fourier spectrum analysis) = f_{res} Hz

Alternatively, the sampling rate may be fixed, for example, r_S = 30 Hz or 60 Hz, while the number of sample points n_S is controlled by T_S : $n_S = r_S T_S$. If $n_S = 2k$, where $k > 0$, then we can detect frequences $1/T_S, 2/T_S, ..., k/T_S$.

4.2 Example

Let n_S = 1000 samples and T_S = 100 seconds. Then

$r_S = n_S/T_S$ = 10 Hz.

$f_N = r_S/2$ = 5 Hz.

f_{max} = 5 Hz.

$t_S = 1/r_S$ = 0.1 second.

Let f = 5 Hz. Then

$p_f = 1/f$ = 0.2 second.

$np_f = T_S f$ = 500 periods.

$ns_f = n_S/np_f$ = 1000 / 500 = 2 samples / period.

$f_{res} = 1/T_S$ = 0.01 Hz.

f_{min} = 0.01 Hz.

Note that the minimal frequency 0.01 Hz has the period of duration 100 seconds, and therefore it has only one period per entire duration of sampling. This frequency is supported by 1000 samples, and it is the smallest frequency computable (detectable) with Fourier transform methods. Therefore, the maximal resolution of its Fourier spectrum consists of frequencies: 0, 0.01, 0.02, ..., 5.0.

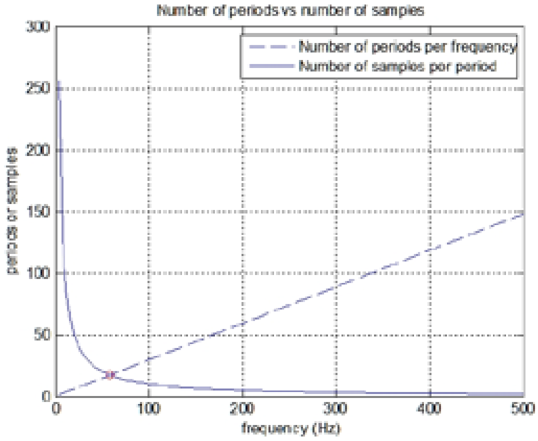

Fig. 3 Number of periods and number of samples

4.3 Frequency limitations

According to the Nyquist Theorem, we can sample and then estimate only those frequencies f which do not exceed f_N . In other words, we need at least 2 samples per period of a computable frequency. The minimal *value* of n_S is 2, where the only detectable frequency is 1 Hz provided that T_S = 1 second. To have more samples is better, however, what we get is usually what we get, and there is no way to get more. The FFT methods also impose the limitation that $f \geq 1/T_S$, or equivalently, we have for the maximum period $p_f \leq T_S$. Therefore we must have at least one period per duration of the data collection. So, we get lower and upper bounds for f

$$1/T_S \leq f \leq (n_S/2)/T_S = r_S/2.$$

In many applications, one period of a sinusoid does not indicate that this period will repeat many times to create a periodic event (phenomenon). I believe, as some other researchers do, that the sampling process must support at least 3 periods of the frequency. To detect smaller frequencies than $1/T_S$ or even $3/T_S$, we should either increase the observation time T_S or use different methods than Fourier spectral methods. Note that increasing T_S decreases the Nyquist frequency when n_S is fixed.

These limits are two extremes: either we have one period stretching over all time T_S and containing all n_S samples or only only 2 samples per each period for all $n_S/2$ periods. The number of periods of a frequency f is $np_f = T_S f$ and changes from 1 to $n_S/2$ as f increases. The number of samples per period of a frequency f is $ns_f = n_S/np_f = r_S/f$, and changes from n_S down to 2 as f increases. These two curves behave like the Supply and Demand Curves in the economic model of price determination in a market (see Figure 3). It is easy to calculate that the equilibrium point is at the frequency $f = \sqrt{ns}/Ts$, and this happens when $np_f = ns_f = \sqrt{ns}$

However, the minimal frequency $f_{\min}$, and the step in frequency scale of the spectrum f_{res} are usually slightly different from $1/T_S$. This happens when the number of samples n_S is not a power of 2. The processing of the number of points which are powers of 2 is required by the recursive formulas in the Fast Fourier Transform algorithm. Let n_{fft} be the length of the Fast

Fourier Transform used in the processing (this number is always a power of 2). For example, if n_S = 296, then n_{fft} can be chosen to be 256 or 512. Now, the formula for the minimal frequency and frequency resolution becomes

$$f_{min} = f_{res} = r_S/n_{fft} = (1/T_S)(n_S/\, n_{fft}).$$

Clearly, if $n_S = n_{fft}$, then $f_{min} = 1/T_S$.

4.4 Signal components

We try to deal with the signal as if it was made of 3 components: the trend, the waves (sinusoids) and the noise, leaving aside the component which is periodic, but with irregular periods. We think about trend as a polynomial, which is clearly a non- periodic component. The polynomial must be of a low degree, as is commonly used in non-linear (polynomial) regression. In the our applications, it was sufficient to deal with a linear trend. The Matlab function detrend removes the best straight-line fit linear trend from the data. Each wave (sinusoid) of a digital signal representing time series is characterized by the frequency, the phase and the amplitude

$$x(t) = A\sin(2\pi Bt + C)$$

The signal may contains several kind of noises, and not all of them additive. Noises may look like aperiodic multi-waves and their spectra may have many spikes. The effects of noises is usually decreased by filtering of the data. We can filter the signal *with a band-pass filter* to cut out very low and very high frequency noises. However, the periodic part of the signal might also be weakened, because we may not know the band containing the dominant frequency. The Matlab function *pwelch* performs the Power Spectral Density (PSD) estimate via Welch's method. By default of this algorithm, the time series is divided into eight sections with 50% overlap, each section is windowed with a Hamming window, and eight modified periodograms are computed and averaged. This kind of spectrum is heavily doctored, so some people prefer the other extreme: the time series are either zero-padded to the nearest higher power of 2 or truncated to the nearest lower power of 2; then, FFT is taken and its the modulus is squared as in the Matlab demo code. This is the raw PSD.

If some frequencies do not spread from the beginning of data to the end, then we must first determine the subintervals of their duration, because each dominant frequency must have specified the time of beginning and the time of end. This is typical, for example, for road vibrations, which are caused by passing of heavy trucks. In this case we use the Short-Time Fourier Transform (STFT) to determine the peaks and the subintervals, and then we can use the Fourier spectra for these subintervals.

4.5 The peaks in PSD

The next step of the algorithm is finding the local maxima in the power spectrum. The peaks (local maxima) in PSD can be detected with pixel accuracy by finding the indices of the array, or with a sub-pixel accuracy, for example, by fitting a smooth 'hat' over the peak in a small neighborhood. The peaks in PSD have approximately Gaussian curve shapes. The energy is stored not only in the dominant frequency but also in the width of the peak curve. Therefore, some sources recommend to consider a narrow bandwidth of dominant frequency, and define it as the width corresponding to frequencies with the amplitude $\frac{A}{\sqrt{2}}$, where A is the amplitude of the peak, or use the Full Width at Half Maximum (FWHM) approach. For most applications, there are ready Matlab functions *findpeaks* and *localmax*.

However, in some cases finding the peaks is not good enough. What should be the relative power of the signal component with the dominant frequency? How much dominant should be the dominant frequency? If the entire power spectrum has the shape of a Gaussian distribution, then its peak is not necessarily strong enough to be considered as the dominant frequency. It may depend on the standard deviation of the spectrum; so one can apply some definitions of the signal to noise ratio (SNR). Some projects require that the peak corresponding to the dominant frequency is 30% higher than other values in the power spectrum. One can ask how much energy (in %) is contained in dominant frequency. Seitz and Dyer [11] calculate the probability that a motion is periodic and compute the most likely period. Also, they use the Kolmogorov-Smirnov test to compare the near-periodic sequences. The Matlab function spectrum.psd can put a ribbon around the PSD estimate according to the specified confidence level p. Matlab has 3 functions to test the significance of the statement whether a signal is just a white noise or not: *kstest, lillietest* and *jbtest*.

Sometimes two or more peaks in PSD are too close, so that their separating requires a special attention. Matlab provides the useful function *periodogram* which may help to balance the filtering and sampling control parameters.

When plotting the phases of frequencies under the power spectrum, we can see, that due to the noise, phases may have jumps; in particular, the phase of the dominant frequency may have a jump. Thus, to extract the complete information from the power spectrum, we need to unwrap the phase array. For this phase correction there is a convenient Matlab function *unwrap*.

4.6 Low frequencies

If the dominating frequency is below $f_{\min}$, then we need to apply different methods than spectral analysis. One approach is to use a least squares fit method to approximate the time series by the function

$$x(t) = A + Bt + C\sin(2\pi Dt + E).$$

I tried this when the sample data contained only 3 periods of the dominant frequency or less. When the time interval contained only a half of the period, the results were still good. When I reached below 1/4 of the period, the results were not so accurate due to the presence of white noise in the data. Matlab has the function *nlinfit*, and the Numerical Recipes [9] have the steepest gradient algorithm *amoeba*.

All issues listed above indicate that the validation of extracted dominant frequency (with the amplitude and phase) must contain several factors such as the number of periods in the sample, relative energy in the bandwidth of the dominant frequency, and the amount of white noise in the data.

5 Dominant frequency in ladar data

The data to be analyzed for dominant frequencies were available only at the end of an engineering process [13], which I describe briefly. A moving and also rotating object

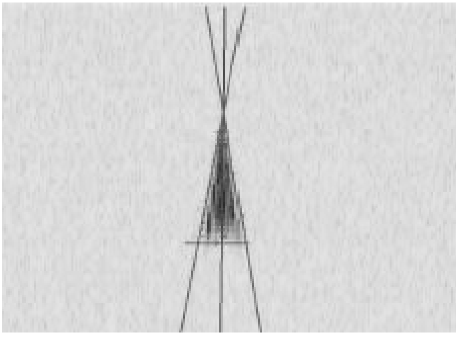

Fig. 4 Feature extraction from a range-Doppler ladar image

was illuminated by an infrared laser, and its image was tracked by a special camera. This arrangement is called laser radar, or ladar. The laser was emitting trains of pulses and the camera recorded the time shifts between emitted and received pulses. Clearly, this was a preparation for the detection of the target range, and also for exploring the Doppler effect, that is, the shift in frequency caused by the component of the target motion in direction of the axis of illumination; actually, in all directions but those perpendicular to the axis of illumination. The resulting image had pixels with two coordinates: the time delay and the phase delay. This image was converted to another image, where the pixel coordinates were the distance and the velocity of the corresponding small portions of the target. This image is called the range-*Doppler image*. The conversion was made by the Short-Time Fourier Transform [1]. So, we got a video, where the target was tracked by a triangle or a quadrangle, and the locations of the vertices were recorded (see Figure 4). Now we got the multidimensional data, which we searched for dominant frequencies.

Not accidentally, ladar imitates the ultrasound imaging system of bats, which they use to catch bugs in fast flights. The only difference is in the variable resolution of images, because when a bat is closing the distance to its prey, which is flapping wings, the resolution is increasing to greater details.

The range-Doppler image is not quite similar to any standard optical image. The optical image has up-and-down and left-and-right orientations, while the range- Doppler image shows what is closer-and-farther and what is moving toward-and-away from the source of illumination. While ladars with their range-Doppler images are typically used for remote sensing applications, like satellites, the Doppler Radars have much wider applications [14].

An interesting part of the project was that I could use a computer simulation to generate a virtual target, where I could enter the dominating frequency as one of input parameters. However, in important test cases, the video data were created by another group, and I did not know the dominating frequency value prior to my analysis. My lucky number was that the dominating frequency, which I extracted from the time series, agreed to all specified decimal points with the frequency known only to the other group (Figure 5)

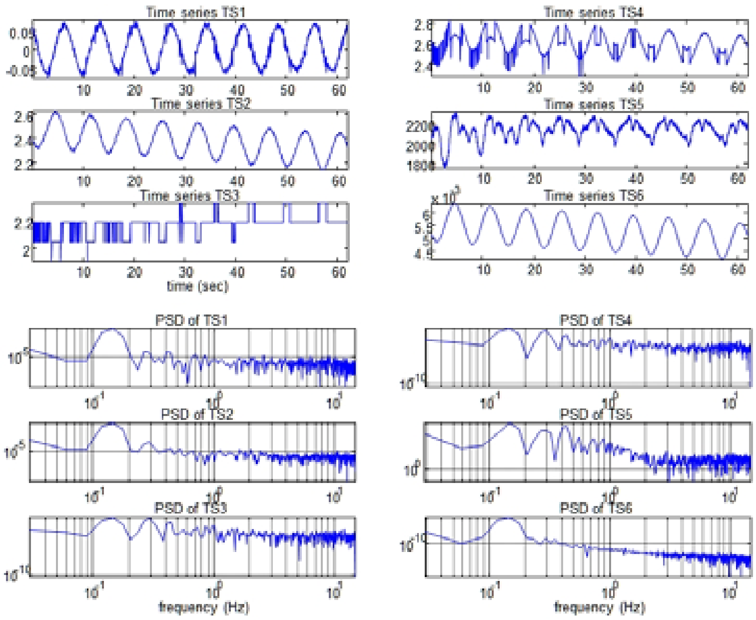

Fig. 5 Six time series and their power spectra

There are many factors influencing the selection of processing methods for time series analysis. Some time series have to be processed in real time in order to trigger a prediction or even a warning, like the monitored vibrations of an overloaded bridge. There is a *wide selection of Power Spectral* Density estimators in Matlab [7]: *pwelch, pmusic, pmtm, pcov, pmcov, pburg, pyulear, periodogram* and the function *spectrum*. The next group are wavelets [5], where the time vs. scale resolution allows to detect events in certain time intervals and being significant in certain scale. The last group of algorithms I want to mention performs special decompositions of the signal such as the Empirical Mode Decomposition and the Independent Components Analysis (cf. [14]). The signal decomposition into multiple latent components using the latent Dirichlet allocation model is studied in [12].

6 Conclusion

The determination of dominant frequency helps to understand the structure of time series, and derive the consequences of the presence and intensity of the dominant frequency, whether it deals with biology, geology, medicine, or other fields of science and engineering. Recognizing the dominant frequency is a part of analysis of data leading to a better prediction, to more accurate diagnosis, and to a better tuned engineering design.

References

Victor C. Chen and Hao Ling: Time-frequency transforms for radar imaging and signal analysis, Artech House, Boston London, 2002.

Benjamin Crowell: Vibrations and Waves, Introductory Physics Textbooks series, Volume 3, 2nd ed., lightandmatter 2000, 2.2 ed., 2006.

R. Cutler and L. S. Davis: Robust Real-Time Periodic Motion Detection, Analysis, and Applications, IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 22, no. 8, pp. 781-796, August 2000.

George C. King: Vibrations and Waves, John Wiley and Sons 2009.

Stéphane Mallat: A Wavelet Tour of Signal Processing, 2nd ed., Academic Press 1999.

D. Mandic, M. Golz, A. Kuh, D. Obradovic and T. Tanaka, (Eds.): Signal Processing Techniques for Knowledge Extraction and Information Fusion, In- formation Technology: Transmission, Processing and Storage, Springer 2008.
Matlab Signal Processing Toolbox (TM) 6, User's Guide, The MathWorks, Inc. 1988-2010 (online only). http://www.mathworks.com.
W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery: Numerical Recipes in C, The Art of Scientific Computing, 2nd ed., Cambridge University Press 1988, 1992.
R. Polana and R. Nelson: Detection and Recognition of Periodic, Non-Rigid Motion Intl Journal of Computer Vision, vol. 23, no. 3, pp. 261-282, June-July 1997.
S. M. Seitz and C.R. Dyer: View-Invariant Analysis of Cyclic Motion, Inter- national Journal of Computer Vision, vol. 25, no. 3, pp. 1-23, 1997.
Matus Telgarsky and John Lafferty: Signal Decomposition using Multiscale Admixture Models, IEEE International Conference on Acoustics, Speech and Signal Processing, 2007 (ICASSP 2007), Vol. II, 449-452.
R. Telgarsky, M. C. Cates, C. Thompson and J. N. Sanders-Reed: High Fidelity Ladar Simulation, Proc. of SPIE, Vol. 5412, April 2004, 194-207.
http://www.wikipedia.org.

GALILEO'S PARADOX[1]

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Abstract:
Galileo refers to demonstration of one of the surprising properties of infinite sets in his revolutionary work "Discourses e Dimostrazioni matematiche intorno à due nuoue Scienze", published in 1638 in Leida. The aim of the article is to show the dispute, which appears in his work in the first chapter. This is apparently contradictory statement about the positive integer. Galileo concluded that the ideas of less, equal, and greater apply to finite sets, but not to infinite sets. In the nineteenth century, using the same methods, Cantor showed that this restriction is not necessary.
Keywords: Galileo Galilei, infinity, paradox, mechanics, one-to-one correspondence
MESC: 01A45, 70-03

1 On the origin of Dialogues

The first thought about writing a book on the mechanics by Galileo appeared in a letter he wrote to his faithful friend Elia Diodati, on March 7, 1634. In the letter Galileo mentioned that he works on a treatise in the field of mechanics devoted to a new topic, based on interesting and useful ideas. It is understandable that his Dialogues on the New Sciences covered many of his earlier results, such as the law of falling bodies in a vacuum, discovered by Galileo before 1610 during his stay in Padua. The Dialogues also contain a supplement about the center of mass, a result from his earlier period. It is difficult to identify the years when Galileo formulated his early results on the strength of materials – in his letters Galileo did not mention these topics, but these results were hardly fully known when Galileo wrote the treatise. The Dialogues are not written as a literary work. When Galileo discovered the laws of nature, he was always trying to do so via finding and formulating direct and simple principles. He would not settle for mere describing the facts as stated in E. Mach2, but he was searching for internal coherence of a more complex a system.

The book „Discorsi e Dimostrazioni matematiche intorno ŕ due nuoue Scienze“ published in 1638 consists of four chapters, written in the form of a dialogue between three friends. The three people are talking about scientific topics and present different views: Salviati is a modern researcher, Simplicio is a traditional scholar, and Sagredo is trying to mediate between the two opposite directions dealing with technical and economic aspects of the new sciences. Filippo Salviati from Florence (1583-1614) and Giovanni Francesco Sagredo (1571-1620) from Venice were intimate friends of Galileo who died before the publication of this work and, in this way,

[1] Article has benefited from the project **KEGA 003KU-4/2013** *Využitie pôvodných matematických demonštrácií a fyzikálnych pokusov, ktoré použil Galileo Galilei v mechanike a pohybe telies vo vyučovaní na základných a stredných školách.*

Galileo pays them a tribute. Simplicio is a fictitious person, whose name recalls the famous commentator of Aristotle, Simplicius of Cilicia. The present paper is based on the Leida edition of the book. The story is located in Venice and comprises four days of dialogues.

The first day deals with the resistance of solids, cutting resistance, resistance to compression, the structure of materials, the existence of a vacuum, floating bodies, parabolic mirror optics, the speed of light, free fall of different masses, the falling of bodies in a vacuum and air, oscillations of a pendulum, the acoustics and music. The second day is devoted to material flexibility and balanced strength. The third day comprises the principles explaining the dynamics of the uniformly linear motion and uniformly accelerated motion, derived from the equations of motion of falling bodies and a mechanical vibration of the pendulum (principle of isochronisms - important for the measurement of time). Experiments deal with metal balls on an inclined plane and related geometrical and mechanical demonstrations. The fourth day deals with missiles; it is the first scientific theory of moving bodies in the two-dimensional space used to explain trajectories of missiles. Galileo shows that the missile follows a parabolic trajectory and demonstrates the principle of composition of movements. At the end of the fourth chapter, there is a geometric table for the calculations of distances to be reached by missiles.

2 Galileo’s paradox

Galileo in his book on page 22 shows the paradox, which refers to infinity. He points out that the terms "equals", "greater than" and "less than" do not apply to an infinite value. Because line segment contains infinitely many points, so the longer line segment would have to contain more points than the previous one and therefore more than an infinite number of points, and it is not possible. That paradox is Salviati using regular hexagon.

Salviati: About G as a center describe an equiangular and equilateral polygon of any number of sides, say the hexagon ABCDEF. (see Figure 1)

Similar to this and concentric with it, describe another smaller one which we shall call HIKLMN. Prolong the side AB, of the larger hexagon, indefinitely toward S; in like manner prolong the corresponding side HI of the smaller hexagon, in the same direction, so that the line HT is parallel to AS; and through the center draw the line GV parallel to the other two. This done, imagine the larger polygon to roll upon the line AS, carrying with it the smaller polygon. It is evident that, if the point B, the end of the side AB, remains fixed at the beginning of the rotation, the point A will rise and the point C will fall describing the arc CQ until the side BC coincides with the line BQ, equal to BC. But during this rotation the point I, on the smaller polygon, will rise above the line IT because IB is oblique to AS; and it will not again return to the line IT until the point C shall have reached the position Q. The point I, having described the arc IO above the line HT, will reach the position O at the same time the side IK assumes the position OP; but in the meantime the center G has traversed a path above GV and does not return to it until it has completed the arc GC. This step

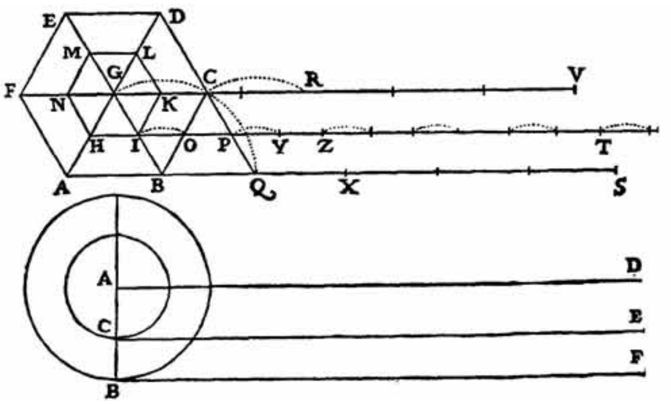

Fig. 1

having been taken, the larger polygon has been brought to rest with its side BC coinciding with the line BQ while the side IK of the smaller polygon has been made to coincide with the line OP, having passed over the portion IO without touching it; also the center G will have reached the position C after having traversed all its course above the parallel line GV. And finally the entire figure will assume a position similar to the first, so that if we continue the rotation and come to the next step, the side DC of the larger polygon will coincide with the portion QX and the side KL of the smaller polygon, having first skipped the arc PY, will fall on YZ, while the center still keeping above the line GV will return to it at R after having jumped the interval CR. At the end of one complete rotation the larger polygon will have traced upon the line AS, without break, six lines together equal to its perimeter; the lesser polygon will likewise have imprinted six lines equal to its perimeter, but separated by the interposition of five arcs, whose chords represent the parts of HT not touched by the polygon: the center G never reaches the line GV except at six points. From this it is clear that the space traversed by the smaller polygon is almost equal to that traversed by the larger, that is, the line HT approximates the line AS, differing from it only by the length of one chord of one of these arcs, provided we understand the line HT to include the five skipped arcs.

Now this exposition which I have given in the case of these hexagons must be understood to be applicable to all other polygons, whatever the number of sides, provided only they are similar, concentric, and rigidly connected, so that when the greater one rotates the lesser will also turn however small it may be. You must also understand that the lines described by these two are nearly equal provided we include in the space traversed by the smaller one the intervals which are not touched by any part of the perimeter of this smaller polygon.

Let a large polygon of, say, one thousand sides make one complete rotation and thus lay off a line equal to its perimeter; at the same time the small one will pass over an approximately equal distance, made up of a thousand small portions each equal to one of its sides, but interrupted by a thousand spaces which, in contrast with the portions that coincide with the sides of the polygon, we may call empty. So far the matter is free from difficulty or doubt.

But now suppose that about any center, say A, we describe two concentric and rigidly connected circles; and suppose that from the points C and B, on their radii, there are drawn the tangents CE and BF and that through the center A the line AD is drawn parallel to them, then if the large circle makes one complete rotation along the line BF, equal not only to its circumference but also to the other two lines CE and AD, tell me what the smaller circle will do and also what the center will do. As to the center it will certainly traverse and touch the entire line AD while the circumference of the smaller circle will have measured off by its points of contact the entire line CE, just as was done by the above mentioned polygons. The only difference is that the line HT was not at every point in contact with the perimeter of the smaller polygon, but there were left untouched as many vacant spaces as there were spaces coinciding with the sides. But here in the case of the circles the circumference of the smaller one never leaves the line CE, so that no part of the latter is left untouched, nor is there ever a time when some point on the circle is not in contact with the straight line. How now can the smaller circle traverse a length greater than its circumference unless it go by jumps?

Sagredo: It seems to me that one may say that just as the center of the circle, by itself, carried along the line AD is constantly in contact with it, although it is only a single point, so the points on the circumference of the smaller circle, carried along by the motion of the larger circle, would slide over some small parts of the line CE.

Salviati: There are two reasons why this cannot happen. First because there is no ground for thinking that one point of contact, such as that at C, rather than another, should slip over certain portions of the line CE. But if such sliding along CE did occur they would be infinite in number since the points of contact are infinite in number: an infinite number of finite slips will however make an infinitely long line, while as a matter of fact the line CE is finite. The other reason is that as the greater circle, in its rotation, changes its point of contact continuously the lesser circle must do the same because B is the only point from which a straight line can be drawn to A and pass through C. Accordingly the small circle must change its point of contact whenever the large one changes: no point of the small circle touches the straight line CE in more than one point. Not only so, but even in the rotation of the polygons there was no point on the perimeter of the smaller which coincided with more than one point on the line traversed by that perimeter; this is at once clear when you remember that the line IK is parallel to BC and that therefore IK will remain above IP until BC coincides with BQ, and that IK will not lie upon IP except at the very instant when BC occupies the position BQ; at this instant the entire line IK coincides with OP and immediately afterwards rises above it.

Sagredo: This is a very intricate matter. I see no solution. Pray explain it to us.

Salviati: Let us return to the consideration of the above mentioned polygons whose behavior we already understand. Now in the case of polygons with 100000 sides, the line traversed by the perimeter of the greater, i. e., the line laid down by its 100000 sides one after another, is equal to the line traced out by the 100000 sides of the smaller, provided we include the 100000 vacant spaces interspersed. So in the case of the circles, polygons having an infinitude of sides, the line traversed by the continuously distributed infinitude of sides is in the greater circle equal to the line laid down by the infinitude of sides in the smaller circle but with the exception that these latter alternate with empty spaces; and since the sides are not finite in number, but infinite, so also are the intervening empty spaces not finite but infinite. The line traversed by the larger circle consists then of an infinite number of points which completely fill it; while that which is traced by the smaller circle consists of an infinite number of points which leave empty spaces and only partly fill the line. And here I wish you to observe that after dividing and resolving a line into a finite number of parts, that is, into a number which can be counted, it is not possible to arrange them again into a greater length than that which they occupied when they formed a continuum and were connected without the interposition of as many empty spaces. But if we consider the line resolved into an infinite number of infinitely small and indivisible parts, we shall be able to conceive the line extended indefinitely by the interposition, not of a finite, but of an infinite number of infinitely small indivisible empty spaces.

Now this which has been said concerning simple lines must be understood to hold also in the case of surfaces and solid bodies, it being assumed that they are made up of an infinite, not a finite, number of atoms. Such a body once divided into a finite number of parts it is impossible to reassemble them so as to occupy more space than before unless we interpose a finite number of empty spaces, that is to say, spaces free from the substance of which the solid is made. But if we imagine the body, by some extreme and final analysis, resolved into its primary elements, infinite in number, then we shall be able to think of them as indefinitely extended in space, not by the interposition of a finite, but of an infinite number of empty spaces. Thus one can easily imagine a small ball of gold expanded into a very large space without the introduction of a finite number of empty spaces, always provided the gold is made up of an infinite number of indivisible parts.

As shown in this excerpt, the possibility of establishing a correspondence between two infinite collections. There was the first historical document on the subject of infinite aggregates.

3 One-to one correspondence

Galileo shows that the longer and shorter line segment has the same number of points as there is between them bijective mapping (see Figure 2).

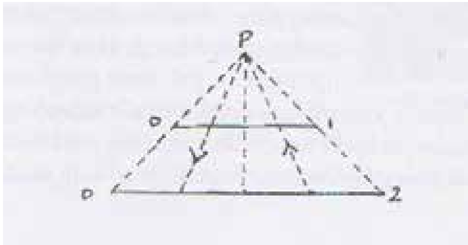

Fig. 3

The picture shows mapping a smaller interval (0,1) to larger (0.2) of point P. Each point in line segment it corresponds to exactly one point in the second line segment and reverse. Even we could even more. As we bent: interval in a semicircle below draw the real line (see Figure 3). Each point of an open interval will correspondence uniquely real numbers, each real number uniquely specified point of time. Thus cardinality set of all real numbers is the same as the cardinality of any open interval.

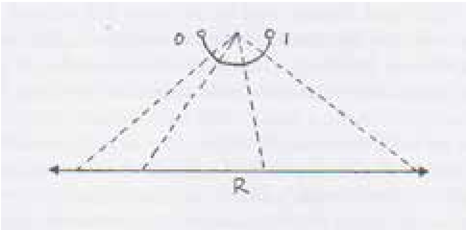

Fig. 2

The aim was to show the mistake of Aristotle's assertion that section must always be smaller than the aggregate. Even clearer that shows an example and shows that the natural numbers is as much as their squares.

Simplicio: A squared number is one which results from the multiplication of another number by itself; thus 4, 9, etc., are squared numbers which come from multiplying 2, 3, etc., by themselves.

Salviati: Very well; and you also know that just as the products are called squares so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are not squares. Therefore if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not?

Simplicio: Most certainly.

Salviati: If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.

Simplicio: Precisely so.

Salviati: But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is a root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers. Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10000, we find only 1/100 part to be squares; and up to a million only 1/1000 part; on the other hand in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers all taken together.

Sagredo: What then must one conclude under these circumstances?

Galileo was the first man in history, that we cannot work the same with infinite and finite sets. Because, it seems in this case that on the one hand, the numbers of squares as well as many integers, but on the other hand, there are clearly less than the integers. For what are we going to continue as the number of square there are "thinner".

Since each number has clearly given the square and the number of square of each corresponds to a one integers must be integers as much as their squares. Still, however, Galileo was wrong in thinking that the points on the line segment is the same as integers.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes “equal,” “greater,” and “less,” are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number. Or if I had replied to him that the points in one line were equal in number to the squares; in another, greater than the totality of numbers; and in the little one, as many as the number of cubes, might I not, indeed, have satisfied him by thus placing more points in one line than in another and yet maintaining an infinite number in each? So much for the first difficulty.

4 Conclusion

The paradox of Galileo evidently left no impression on his contemporaries. For two hundred years nothing was contributed to the problem. Then in 1820 there appeared a small tract in German by one Bolzano, entitled “The Paradoxes of the Infinite”. This attracted little attention, so little indeed, that when fifty years later the theory of aggregates became the topic of the day, few mathematicians knew who the man was.

Today Bolzano’s contributions are of a purely historical interest. While it is true that he was the first to broach the question of the actually infinite he did not go far enough. Due honor must be given the man for creating the all-important concept of the power of an aggregate of which I shall shortly. The modern theory of aggregates begins with Georg Cantor. His essay, which laid the foundation of this new branch of mathematics, appeared in 1883 under title “On Linear Aggregates”. This essay was the first to deal with the actually infinite as with a definite mathematical.

References

GALILEO GALILEI: Discorsi e Dimostrazioni matematiche intorno à due nuoue Scienze. Leida: Elsevier, 1638.

TKAČIK, Š.: On original mathematical demonstrations and physical experiments by Galileo Galilei, In: Proceedings of the Conference on the History of Mathematics and Teaching of Mathematics : Comenius Faculty College of the University of Miskolc : Sárospatak 23-27 May 2012 / ed. Péter Körtesi. - Miskolc : University of Miskolc, 2012. - ISBN 978-963-661-988-6, CD-ROM, p. [1-4].

KAPLAN R., KAPLANOVÁ E.: Umění nekonečna, náš stracený jazyk čísel. Praha: Triton. 2010. ISBN 978-80-7387-245-8

SEVERAL NOTES ABOUT THE HARMONIC SERIES

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Abstract:
The Harmonic Series is one of the best known infinite series in mathematics. It plays an essential role in creation of students' conception about convergence or divergence of series and also about the speed of divergence. The paper presents and analyses some practical examples leading to the harmonic series.

Key words: Harmonic series, Euler's formula
MESC: I30

1 Introduction

The harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

is one of the most celebrated infinite series of mathematics. As a counterexample illustrates, the convergence of terms to zero is not sufficient to the convergence of a series.

There are many ways to prove that the sum of the harmonic series is infinite (see [4]). To determine the N-th partial sum, we usually use the *Euler's formula*

$$\sum_{n=1}^{N} \frac{1}{n} = \ln N + \gamma,$$

where γ is the *Euler's* or the *Euler-Mascheroni's constant*, which value is approximately 0.577216. It is generally a great surprise for the students to find out how inconceivably slow the speed of the divergence is. Even a really efficient computer does not lead us to expect the sum of $+\infty$. Just to illustrate, if we used a computer that sums up 10^{10} elements in a second and the summing had started 13.5 billion years ago (straight after the big bang), we would receive the sum less than 100 today.

2 The harmonic series in a daily life

The following example demonstrates that we meet the harmonic series even in those situations, where we would very likely not expect it at all.

Example:

Let us have n identical cubes (blocks) and we should construct a tower as seen in Fig. 1 (corresponding edges of the cubes are parallel). What is the maximum possible value of H_n so that the tower does not fall down? What is the maximum possible value of H_∞, i.e. if we had a limitless amount of cubes?

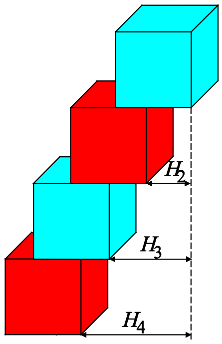

Fig. 1

Solution:

Let us suppose that the cubes have edges of the lengths of 2. In the figures, we will represent just the frontal sides of the cubes and in the descriptions we will number the cubes from above. The most simple example is the one with only two cubes, where it is clear that H_2=1 because the centre of gravity of the first (red) cube is lying above the edge of the second (blue) cube (Fig. 2).

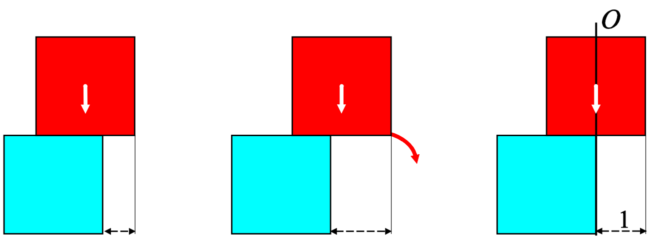

Fig. 2

In the case of three cubes the centre of gravity of the solid created of two upper cubes has to lie above the edge of the third cube again, i.e. on axis *O*. This condition is fulfilled as long as the "mass" to the left of axis *O* is equal to the "mass" to the right of axis *O* (Fig. 3 a, b). It means that the second cube has to be moved (considering the third cube) by one fourth of the cube, i.e. by $\frac{1}{2}$.

Then, evidently, $H_3 = 1 + \frac{1}{2} = \frac{3}{2}$.

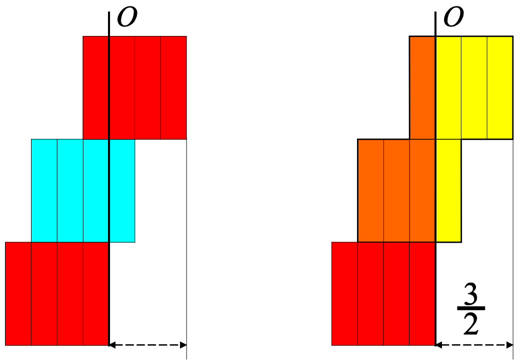

Fig. 3 *a)* *b)*

In the case of four cubes the centre of gravity of the solid created of three upper cubes is lying on axis *O*, that is the "mass" to the left of axis *O* is equal to the "mass" to the right of axis *O* (Fig. 4 a, b). It is necessary that the third cube was moved (considering the fourth cube) by one sixth of the cube, i.e. by $\frac{1}{3}$. So, evidently, $H_4 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

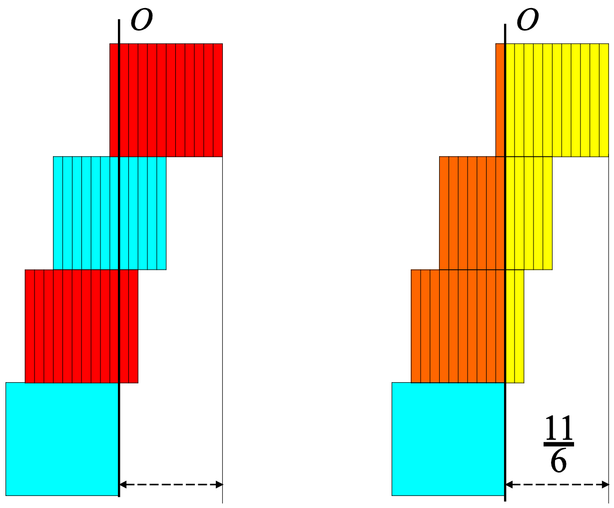

Fig. 4 *a)* *b)*

Generally speaking, let us have a balanced system of *n* cubes and let us add the (*n*+1)-th cube underneath. We come out of the formula for calculating the *x*-coordinate of the centre of gravity of a system of *n* points (coordinates $x_1, x_2, \ldots, x_n$ and with weights $m_1, m_2, \ldots, m_n$).

$$x_T = \frac{m_1 \cdot x_1 + m_2 \cdot x_2 + \ldots + m_n \cdot x_n}{m_1 + m_2 + \ldots + m_n}$$

We suppose to have *n* cubes balanced (Fig. 5). Now, we can look at the original system of *n* cubes as it was a point with "weight" *n* and the centre of gravity is above the edge of the lower cube, i.e. at point 1. The centre of gravity of the lower cube is at point 0.

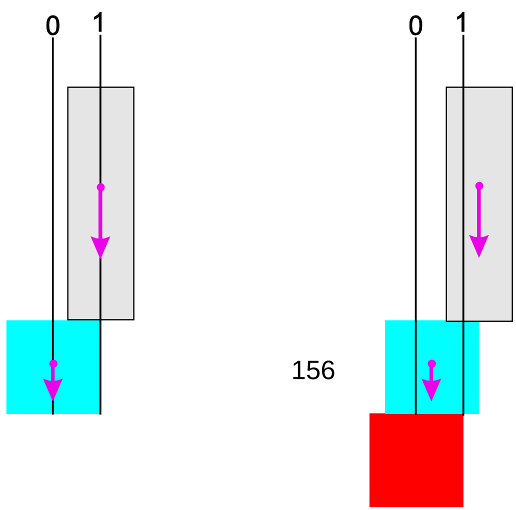

Fig. 5

Then we ,,insert" the red cube underneath, i.e. we move the whole system by h_{n+1} to the right so that the x-coordinate of the centre of gravity of a system of (n+1) points was at point 1 again. We get the following condition

$$1 = \frac{1 \cdot h_{n+1} + n \cdot (1 + h_{n+1})}{1 + n},$$

that is

$$h_{n+1} = \frac{1}{n+1}.$$

Then, we get

$$H_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}.$$

3 Conclusion

We can see that the particular moves of the cubes generate the harmonic sequence and the searched sequence H_n is the sequence of partial sums of the harmonic series. That is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n - 1}$$

and

$$H_\infty = \infty.$$

That means, for the tower in Fig. 1, that it is possible, in theory, to build infinitely many cubes on each other (where every cube is shifted considering the previous cube) and the tower would still be balanced.

References

ASH, J. M. Neither a worst convergent series nor a best divergent series exists. College Mathematics Journal. 1997, 28(4), 296-297.

CUSUMANO, A. The harmonic series diverges. American Mathematical Monthly. 1998, 105(7), 608.

FLERON, J. F. Gabriel's wedding cake. College Mathematics Journal. 1999, 30(1), 35-38.

KIFOWIT, S. J. a T. A. STAMPS. The Harmonic Series Diverges Again and Again. The AMATYC Review. 2006, 27(2), 31-43.

DYNAMIC VISUALIZATIONS IN EDUCATION OF MATHEMATICS

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Abstract:
In the paper we discuss some of the new challenges of ICT aided education and we present some examples of utilization of dynamic mathematical software GeoGebra in teaching basic courses of university mathematics and geometry at technical university.
Key words: Visualization, dynamic mathematics, interactive worksheets, applets
MESC: U70, N80

1 Introduction

Mathematics is one of those subjects, in which teaching requires illustrations of abstract theoretical concepts and relations in a suitable way on examples, with aim to help students in better understanding of connections and enable them to attain better insight to the presented problems. Dynamic mathematical programmes are ideal educational tools enabling visualization of mathematical entities in the form of dynamic models. Direct interactive manipulations with models offer possibilities to users of heuristic approach in acquiring knowledge, while learning of theoretical data is directly connected with their practical application in the respective dynamic model. Generation of model itself and also work with it during study require a new attitude to the role of teacher and student in the educational process, change of the form of educational environment and contents of the educational process.

Research in the domain of cognitive psychology [1] shows that human brain stores knowledge in two forms: in the graphical form as images, or in the form of words. It has been documented that methods of education, which are oriented to development of both of these forms of knowledge acquisition, in a considerable way influence quality and depth of understanding and sustainability of acquired knowledge. Direct inclusion of learners into the development and utilisation of non-linguistic representations within the process of learning considerably stimulates and increases brain activity. This further leads to the development of cognitive connections, which consequently foster knowledge and deepen understanding of basic principles and concepts. Manipulative techniques and tools are concrete or symbolic artefacts that students directly use while obtaining new pieces of knowledge. These powerful didactical tools enable active, hands-on explorative methods and heuristic investigation of abstract rules and relations. Research results prove that computer aided manipulative techniques and tools are even more effective than usage of physical objects, i.e. three-dimensional models, because they can dynamically connect several possible representations and interpretations of studied concocts and relations.

Dynamic working sheets developed in the dynamic mathematical program GeoGebra provide users with possibilities to define dynamical mathematical objects, as e.g. graphs of functions, polygons, curve segments, etc. and interactively manipulate with them. These mathematical objects are real objects of the GeoGebra environment virtual platform, and we can therefore speak about continuous dynamic interaction between these objects and users. Software GeoGebra can be regarded as a manual equipment detecting motion of slider and consequent changes of the depicted objects, as presented also by Karadag and McDougall in [2].

2 Visualization in Maths using dynamic applets

Visualization of basic concepts, relations and dependencies is undoubtedly an integral part of the educational in mathematics. It works with models of these abstract entities, which can appear in various forms, for instance as three-dimensional real models, different didactic tools, images, maps and graphs, films and video presentations, or as concrete activities during the teaching process and practical modelling, i.e. development of models in some way visualizing particular concept or relation. Today, applications utilizing latest information and communication technologies should be included unarguably. Dynamic models play one of the formative roles in the process of knowledge acquisition, as they stimulate cognitive processes and enable development of life, interactively manipulative cognitive connections.

GeoGebra is an intuitive user-friendly application suitable for all users without any specific needs and skills in information technology. This free software, available from the webpage [3], can serve for development of dynamic visualizations and applications that can be presented directly on the web as dynamic web pages in the form of html files or executable Java applets. GeoGebra is available in more than 50 languages accessible on a click from the menu, while complete construction protocol appears directly in the selected language. In this way, direct share of developed materials is enabled in the wide international context. A rich database is available in the user‘s forum GeoGebraWiki.

GeoGebra Works in two windows, geometric Euclidean plane with Cartesian coordinate system and algebraic for mathematical expressions, while two kinds of commands are accepted - geometric (plot of points, lines and line segments, polygons, conic sections) and algebraic (plots from equations of curves and graphs of functions). Tangent to algebraic or transcendent curve in given point is possible as included command, while plot and equation appear simultaneously in the respective windows. Plot of a set of points determined by a simple geometric condition is presented in Fig. 1., where red curves are two parts of an equidistant of a plane curve plotted in blue in the distance d. Value of distance d can be dynamically changed using the slider.

Dynamic constructions and manipulations are the most powerful tools of software GeoGebra. Users can create dynamic working sheets for demonstration of various rules and properties generally valid for specific geometric figures with possibility to change their form dynamically. Properties of tangent line to parabola and points symmetric to focus of parabola with respect to its tangent lines are illustrated in Fig. 2. Construction is saved in the construction protocol, which can be presented on demand from menu, and can be repeated step by step. It appears automatically in the selected language of GeoGebra, no translation is therefore necessary. Construction can be initiated from the beginning using buttons on the navigation panel, and displayed step by step or automatically, with the predefined timing.

Geometric interpretation of derivative of function in given point is a classical example of visualisation in mathematics. This basic concept can be presented in an interactive java applet, where graph of investigated function, tangent line to this graph in a given point and slope of this tangent line are presented, see Fig. 3. Dynamic illustration of the changing slope of tangent line in correlation to the value of the function first derivative in the respective point is a key concept for investigation of the function behaviour. Another concept, limit of a function, is presented in a dynamic definition, in which sliders are used to determine arbitrary value of ▱ while corresponding value of ▱can be thus found. Arbitrary function formula can be inserted into the input window, which makes this demonstration applet a universal modelling tool presenting concept of limit, and partially also continuity, Fig. 4.

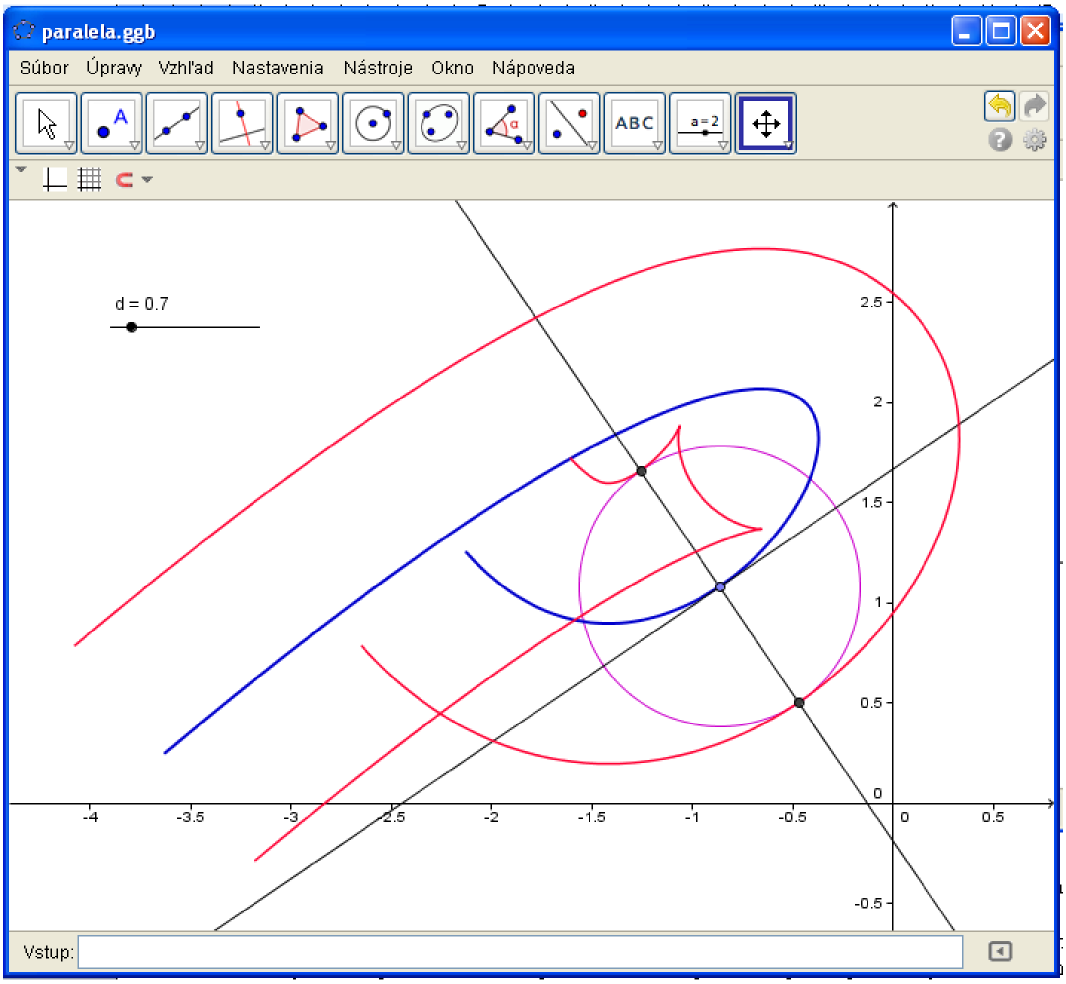

Fig. 1 Equidistant to a planar curve

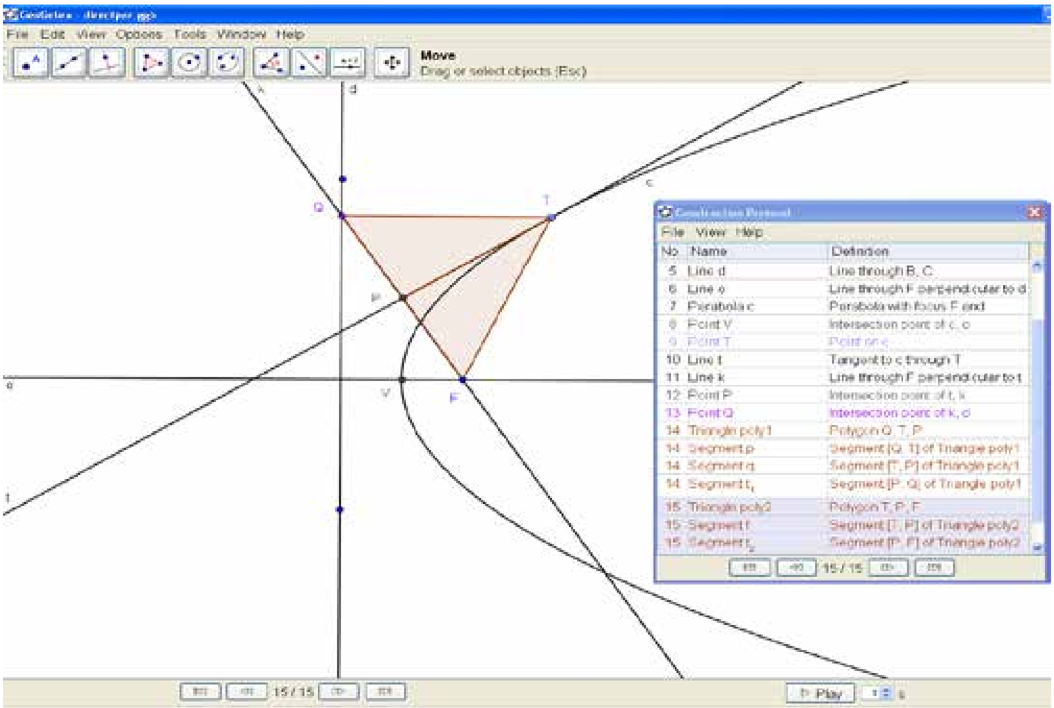

Fig. 2. Tangent line to parabola

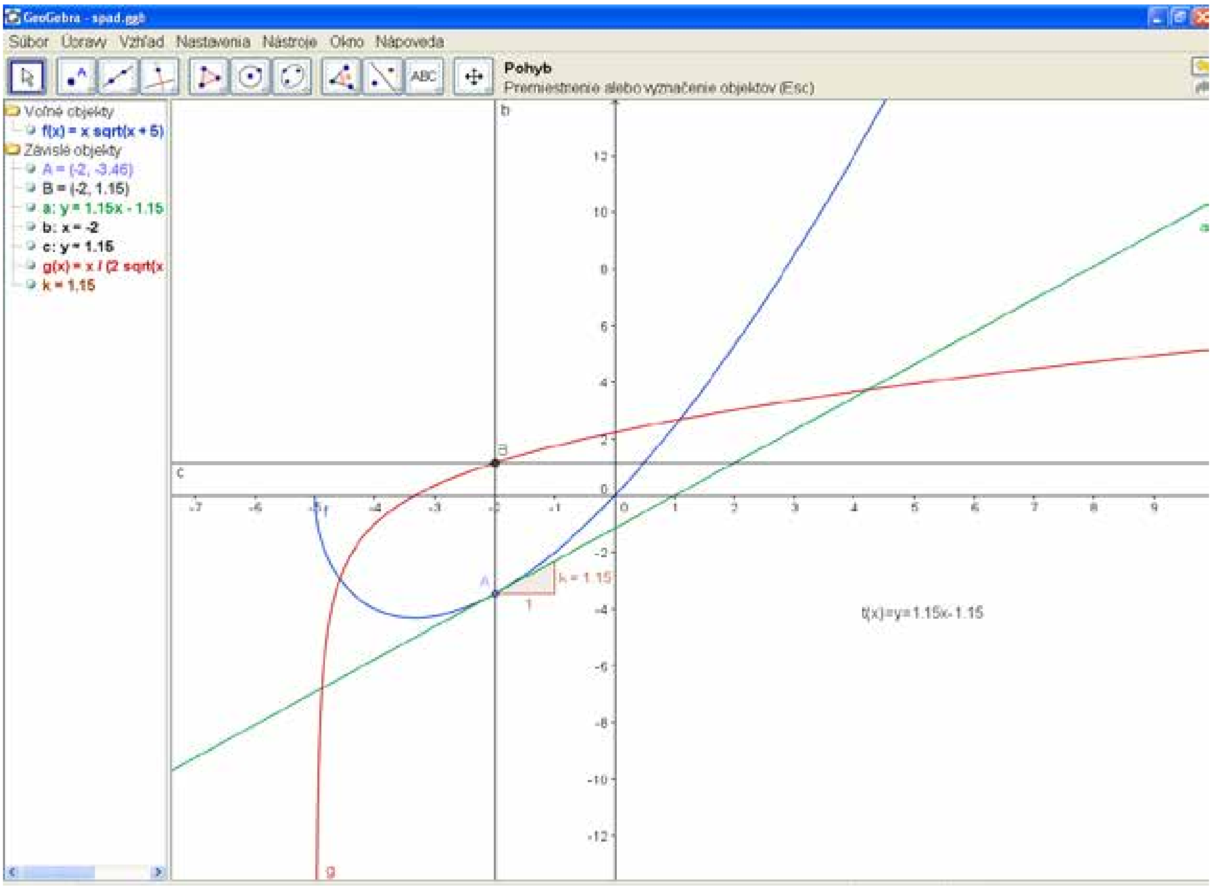

Fig. 3. Derivative of function in the point

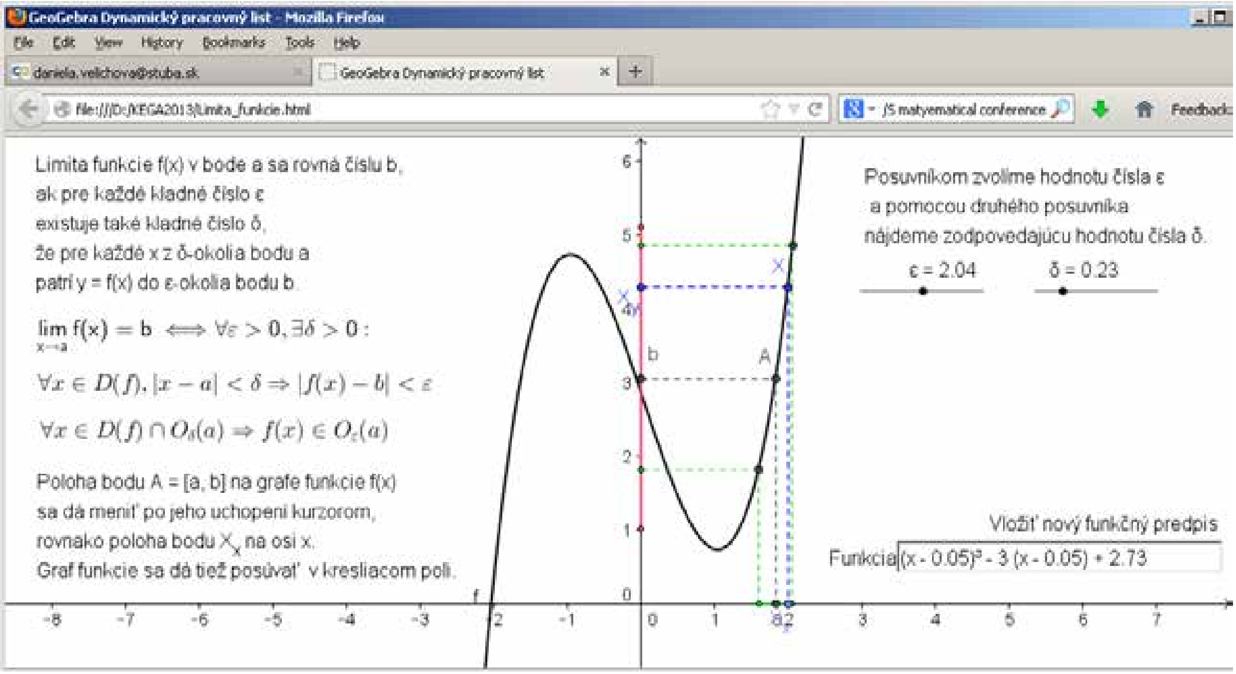

Fig. 4. Limit of function

In addition to interactive modelling in the plane, one can prepare dynamic applets for visualisation of 3D scenes by means of projection equations. Bézier approximation cubic is presented in Fig. 5. Triples of Cartesian coordinates of basic points P_0, P_1, P_2, P_3 form the input data, and can be interactively changed by sliders. Program automatically recalculates coordinates of points of axonometric view (plotted in black colour) and points of axonometric ground view (plotted in blue colour) of the modelled curve. Basic polygon is depicted in red in both views, similarly as views of coordinate axes. Tangent vector in the curve point is plotted in the arbitrary point $P(t)$, in green colour in axonometric view and blue in axonometric ground view. Point P is uniquely determined by a curvilinear coordinate, value of parameter t, which can be selected in a unique interval using slider. Equations of curve are presented in the geometric window, while any change of the input data, i.e. any coordinate of any vertex of basic polygon, is automatically interactively reflected in the change of the form of the curve equation. Two angles, a – azimuth and e – elevation, which can be also determined interactively by sliders, determine actual axonometric projection. Changing their values, users can follow curve shape from different views. Azimuth represents angle of revolution about axis z, while elevation defines angle of revolution about axis x, i.e. declination of coordinate plane xy with respect to projection plane. Combination of values of the two angles of revolution about coordinate axes can produce an arbitrary orthographic view of the 3D scene to one plane, e. g. values $a = e = 0°$ represent front view, values $a = 90°$, $e = 0°$ side view, $a = 0°$, $e = 90°$ ground view, $a = 45°$, $e = 45°$ isometric view.

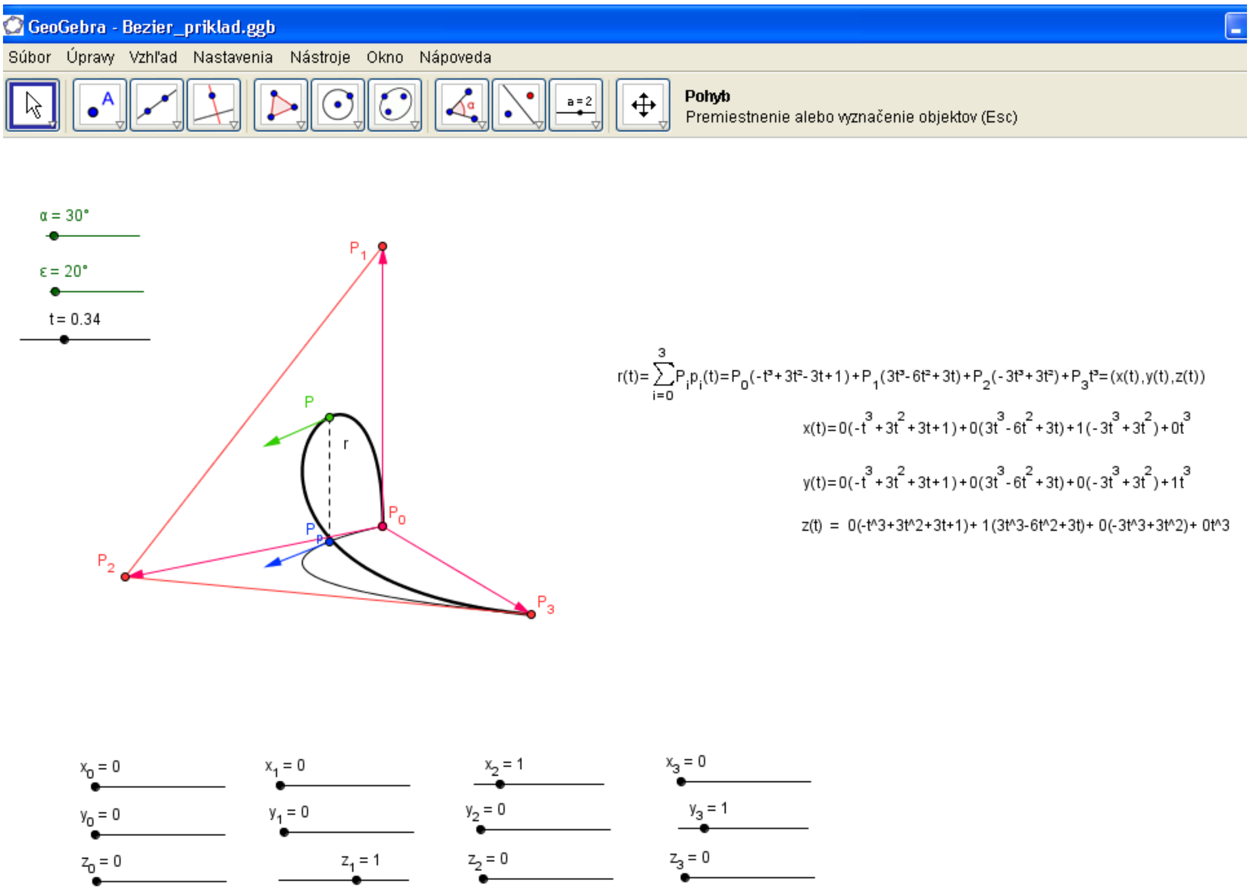

Fig. 5. Bézier cubic in space

3 Conclusion

Visualization can be regarded as a certain form of application, therefore development of visual models is also a kind of evaluation of the knowledge depth and level of understanding of the presented concept, fostering acquired knowledge and its usage, transfer of knowledge into a different context. Development of dynamic models is also an inspiration how to utilize information technologies meaningfully in the role of didactic tool, which can not only attract learners, but also enable them to realize their own creative work. Both subjects of the educational process act in this didactic situation more as equal partners, not as it is usual in the classical forms of didactic situations, where the role of teachers is active presentation of new facts and data, while role of learners is usually passive, just receiving presented facts. Active participation of students in the process of education in an interesting form can contribute to a better understanding and definitely to a more positive approach to learning itself, which becomes more a discovery of dependencies and investigation of activities and processes than memorising of a huge amount of incomprehensible facts and data which are not connected. Dynamics opens way to discover connections, and to understand mutual dependencies, which is often more important than a detailed fragmented knowledge.

Finally, it is necessary to admit that a pure ability to plot and graphically interpret mathematical object itself does not mean any full understanding of the determining mathematical relations geometrically represented by the generated visual image. Dynamic work with these objects is necessary for achievement of a full understanding and for development of abilities to utilise technologies for analysis and solution of problems. This is fully consistent with the general understanding of mathematical knowledge as sustainably increasing ability to utilise various representations of mathematical concepts in different contexts, exchange them dynamically according to actual needs, and be able to illustrate and apply these concepts correctly.

References

MORENO, A., ARMELLA, L., HEGEDUS, S. J., KAPUT, J. J.: From static to dynamic mathematics: Historical and representational perspectives. In: Educational Studies in Mathematics, Vol. 68: 99-111, 2008.

KARADAG, Z., McDOUGALL, D.: Dynamic worksheets: visual learning with the guidance of Polya, In: MSOR Connections, Vol. 9, No 2, May – July 2009.

GeoGebra webpage. [online]. Available at the address: < www.geogebra.org >.

VELICHOVÁ, D.: Creating Cognitive Connections in Maths with GeoGebra, In: Proceedings of the XXVIII International Colloquium, Brno 2010.

VELICHOVÁ, D. Geometrické modelovanie - matematické základy, SjF STU, Vydavateľstvo STU Bratislava, 2005, ISBN 80-227-2179-4, 140 s.

IMPROVING THE LOGICAL THINKING THROUGH SOLVING OF EINSTEIN´S PUZZLE USING MANIPULATIVES

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Abstract:
The main goal of teaching mathematics is to prepare students to be able to use their acquired knowledge and skills to solve real-life problems. Logical thinking is very crucial when solving real-life problems, therefore we need to continuously develop students´ logical thinking during the learning process through problem solving, for example from the area of recreational mathematics. One of these tasks can be the Einstein´s puzzle: Neighbours. In this article we describe the students´ solutions of this puzzle. Students solved this puzzle using manipulatives – paper houses and cards with details about neighbours. We were interested which of the given information confuse students the most and which of them can be understood in more ways than one.
Key words: Puzzle-based learning. Recreational mathematics. Einstein´s puzzle.
MESC: U60, D40

1 Introduction

The modern teaching methods in mathematics education (for example problem–based learning, project-based learning) place emphasis on solving real-context tasks during the mathematics lessons. Students often need logical thinking for solution of these kinds of problems or tasks. Logical thinking is defined by Chytrý[1] (2013) as *a process in which the individual looks back from the content of particular statements, and consistently employs particular inferences, so that he arrives at a correct conclusion. The indisputable partial steps of the process create a link between the assumptions and the conclusion through a chain of these inferences*. Each student has logical thinking developed to different level.

Logic and logical thinking is important for everybody who needs to use her/his natural language as their daily tool. Gahér (2003) claims that not only high school or university students need to use the natural language sufficiently precise and logically correct, but also all those whose quality of performance depends on the accuracy and adequacy of the statement text capture ideas, where the formulations are unequivocally desired logical consequences.[2]

For example the commentator after one ice-hockey match (the Slovak team lost) said: "*we lost three important points*" – but the formulation was wrong – how could the team loose point which they had not had before?

[1] CHYTRÝ, V. Development of logical thinking using mathematical Games. Ústí nad Labem: UJEP, 2013
[2] GAHÉR, F. Logika pre každého. Bratislava: IRIS, 2003. ISBN 80-89018-54-8

Most of the students have never learnt how to think about solving problems (Michalewicz Z., Michalewicz, M., 2008). For homework we mostly give students tasks or problems placed at the end of the chapter in textbook that refer to a particular solution method. In real life, all we have is the problem itself, and all the knowledge and experience we have accumulated over time (Tomasi, 2006)

Other important part of solving problems or tasks is motivating students to do this work. Bruckman (1999) claimed "*a useful exercise is to think back to a moment when you were inspired by a new idea – when it really seemed magical*". We thought about our inspiration in mathematics – part of it was solution of puzzles focused (at least a bit) on mathematics. The mathematically oriented puzzles take part of recreational mathematics.

1. 1 Recreational mathematics

As Hankin[3] (2013) has written, recreational mathematics is easier to recognize than define. We can describe recreational mathematics as a part of mathematics which refers to mathematics carried out for recreation, self-education and self-entertainment, rather than as a fully serious professional activity. It often involves mathematical puzzles and games. Many tasks in this field require no knowledge of advanced mathematics and recreational mathematics often attracts the curiosity of amateur mathematicians, inspiring their further study of the subject.

Kulkarni (2012) declares that "*students´ motivation for learning mathematical problem solving can be improved dramatically by recreational mathematics*". He has used puzzles to teach creative problem solving to students over the last ten years. He claims that his students found math classes a very enjoyable experience[4]. In according to Švecová and Rumanová (2012) that creativity in the mathematics classroom is not just about what pupils do but also what we do as teachers. If we think creatively about mathematical experiences we offer our pupils opportunities to be creative in problem solving. We return to Kulkarni: *Once students begin to love creative math problem solving, they have an activity they can enjoy wherever they are. Then, the joy of creative thinking is all they need to motivate themselves to get going on any challenging math problem.*

How recreational mathematics is useful sums up Singmaster (1993) in following points[5]:

- Recreational problems are often the basis of serious mathematics. For example Graph Theory – Euler´s Königsberg bridges problem is a very popular recreational task and the basic of the Graph theory. Although graph theory is not a part of standard curriculum, often simplifies the solving other mathematical problems. We agree with Melušová[6] (2012) that "*graphical representation can facilitate the search for organizational principles to solve problems in combinatorics, but also in solving problems in other areas of mathematics"*. Consequently the study of recreational topics is necessary to understanding the history of many, perhaps most, topics in mathematics.
- Recreational mathematics has frequently turned up ideas of genuine but non-obvious utility. For example the theory of probability and statistics grew from the analysis of gambling bets

[3] HANKIN, R. K. S. Recreational mathematics with R: introducing the magic package. Online: http://cran.r-project.org/web/packages/magic/vignettes/magicpaper.pdf

[4] KULKARNI, D. Enjoying Math: Learning Peoblem Solving with KenKen® Puzzles. California: Pleasanton, 2012. ISBN 9778-0615702438

[5] SINGMASTER, D.: The Unreasonable Utility of Recreational Mathematics. Online http://www.eldar.org/~problemi/singmast/ecmutil.html

[6] PAVLOVIČOVÁ, G. et al.: Experimentujeme v elementárnej matematike. Nitra: UKF, 2012. ISBN 978-80-558-0126-1

to the basis if the insurance industry in the 17th and 18th centuries. The theory of Latin squares began as a recreation but become an important technique in experimental design.

- Recreational mathematics has great pedagogic potential:
- a) It is a treasury of problems which make mathematics fun. These problems are often based on reality, though with enough whimsy so that they have appealed to students and mathematicians for years. They illustrate the idea that “Mathematics is all around you – you only have to look for it”.”
- b) A good problem is worth a thousand exercises – recreational mathematics provide many such problems and almost every problem can be extended or amended. Hence recreational mathematics is also a treasury of problems for student investigations.
- It has a long history, so recreational mathematics is an ideal vehicle for communicating historical and multicultural aspects of mathematics.
- Recreational mathematics is very useful to the historians of mathematics. Recreational problems often are of great age and usually can be clearly recognised, they serve as useful historical markers, tracing the development and transmission of mathematics (and culture in general) in place and time. The Chinese Remainder Theorem and the Cistern problem are excellent examples of this process.

1.2 Puzzle-based learning

Several mathematicians believe that puzzles (usually mathematical puzzles) are quite educational and that we should educate students by incorporating puzzles into various curricula. Michalewicz, Falkner, Sooriamurthi (2011) created and experimented with a new approach – puzzle-based learning, that is aimed at getting students to think about how to frame and solve unstructured problems. Authors´ goal is to motivate students, and also to increase their mathematical awareness and problem-solving skills by discussing a variety of puzzles and their solution strategies.[7].

As we mentioned before problem solving in the real world requires a continuum of learning. The authors (Michalewicz, Falkner & Sooriamurthi) created a system of learning, in which each layer of skills builds upon the layers below it (Figure 1).

[7] MICHALEWICZ, Z. – FALKNER, N. – SOORIAMURTHI, R. Puzzle-Based Learning: an Introduction to Critical Thinking and Problem Solving. In: Decision Line, 2011.

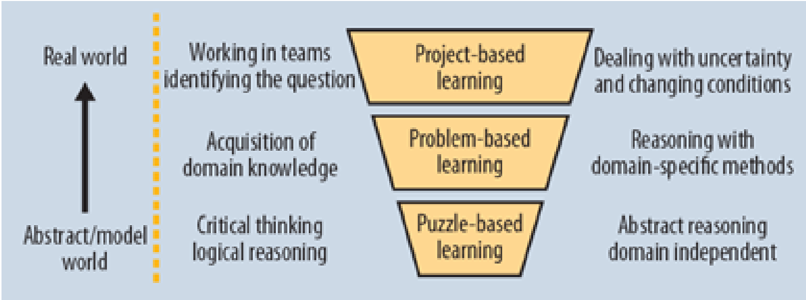

Fig. 1 System of learning and developing of skills needed for problem-solving and critical thinking (Falkner, Sooriamurthi, Michalewicz, 2010)[8]

By authors´ description, problem-based learning requires significant domain knowledge. Project-based learning on the other hand, deals with complex situations in which usually no clearly unique or correct way of proceeding exists. Puzzle based learning focuses on domain-independent critical thinking and abstract reasoning.

Puzzle based learning is in progress, but the first results indicate that students who enrol in that course perceive an improvement in their thinking and general problem-solving skills (Falkner, Sooriamurthi and Michalewicz, 2010).

1.3 Using Manipulatives

Manipulatives are often defined as "*physical objects that are used as teaching tools to engage students in the hands-on learning of mathematics* (Boggan et al. [9], 2010). Manipulatives are suitable for using in all areas of math instruction – teaching number and operations, algebra, geometry, measurement across all grade levels. They can be used to introduce, practice, or remediate a concept. Manipulatives can be almost anything – blocks, shapes, penny or even paper that is cut or folded. They may be store-bought, brought from home and teacher– or student- made.

Sometimes it is necessary to use manipulatives. Pupils and students on some level of mental maturity are unable to understand the content of abstract concepts, respectively relations expressed only through words and mathematical symbols (Vidermanová and Klepancová, 2012).

2 Students' solutions of the einstein´s puzzle

For our experiment we chose the ***Neighbours*** puzzle. This is a Zebra type puzzle, often called Einstein's puzzle or Einstein´s riddle. This puzzle is circulated with the note that Einstein thought up the puzzle as a boy and he said that only 2 % of population could solve it.

The wording of the puzzle is:

> *There are five houses of different colour next to each other on the same road. In each house lives a man of a different nationality. Every man has his favourite drink, his favourite brand of cigarettes, and keeps pets of a particular kind. We have following information about the neighbours:*
>
> *1. The Englishman lives in the red house.*
>
> *2. The Swede keeps dogs.*
>
> *3. The Dane drinks tea.*

[8] MICHALEWICZ, Z. – FALKNER, N. – SOORIAMURTHI, R. Puzzle-Based Learning for Engineering and Computer Science. In: Computer, 2010.

[9] BOGGAN, M. et al. Using manipulatives to teach elementary mathematics. In: Journal of Instructional Pedagogies. ISSN 1941-3394

4. The green house is next to the white house, on the left.
5. The owner of the green house drinks coffee.
6. The Pall Mall smoker keeps birds.
7. The owner of the yellow house smokes Dunhill.
8. The man in the centre house drinks milk.
9. The Norwegian lives in the first house.
10. The Blend smoker has a neighbour who keeps cats.
11. The man who smokes Blue Masters drinks beer.
12. The man who keeps horses lives next to the Dunhill smoker.
13. The German smokes Prince.
14. The Norwegian lives next to the blue house.
15. The Blend smoker has a neighbour who drinks water.
Who keeps fish?

The answer to this puzzle is *German keeps fish*. In our experiment we were interested not only in the answer to the question, but in the whole solution of the puzzle. So under the solution of the puzzle in the following text we will understand the placing of all the given cards.

We assigned this puzzle to students of study program pre-school and primary school teachers education. These students attended a subject called Workshops in mathematics, and part of this subject´s curriculum is development of logical thinking through solving interesting tasks. 81 students participated in this subject in five classes. We used group-work (2 or 3 students in each group, 30 groups together). We prepared some manipulative teaching aids for students (Figure 2) – a row of paper houses and cards with the attributes of occupants (colour of the house, favourite drink, favourite brand of cigarettes, kind of pet; nationalities).

Fig. 2: Manipulatives used in solution of Einstein´s puzzle

We tried to record their solutions, but it was ineffective (noise and little place in classroom, not enough camcorders). Despite all these difficulties we have had the students´ solutions of the puzzle, above all their approaches to solution.

From further experiences with this puzzle we have known that students have problem with left and right side. Common student´s question is: It is on the left in solver perspective or in „houses" perspective? The left side problem doesn´t influence the answer of the question – the only difference in both possibilities (Figure 3 and Figure 4) is that the green house and the white house with all attributes exchange their positions. We modelled on these experiences two expected solutions:

Fig. 3 Solution 1 - with green – white position, the left side in our perspective (front view)

Fig. 4 Solution 2 - with white – green position, the left side in "houses" perspective (back view)

Solution 1 is in our opinion the easier approach – the pair green + white could be only the 4th and the 5th house.

Now we can focus on what have the students done. The group stopped working after teacher´s control. Each group finished only after they had the correct solution. The average solution-time of groups was 27 minutes. The fastest group needed only 10 minutes, but the slowest group needed 67 minutes.

After the analysis of students solutions (teachers´ notes, photos, recordings) we divided the groups to three categories (Table 1).

	Number of groups
solution alone	22
solution with teacher´s help	4
"cheat" solution	4
	30

Tab. 1: Success of solution of groups

To "cheat" section we put the groups which picked the solution up by watching other groups (those groups which were after the teacher´s control).

We were surprised by the solution 3 (Figure 5) and solution 4 (Figure 6). Two groups (not together in the classroom at the same time) put the first house at the end of the row – it is the first house from the other side of row. That was from information 9: *The Norwegian lives in the first house*.

Fig. 5 Solution 3 – first house at the end of the row (white – green position in back view) – one group´s solution

Fig. 6 Solution 4 - – first house at the end of the row (green - white position in front view) – one group´s solution

In general, students solved the puzzle using these approaches:

1. They put the cards with attributes randomly to the houses (8 groups). After it they tried to exchange cards in order to given information. It was impossible for them and they started the solution again, sometimes they started three or four times from the beginning. To this 8 groups belong the four groups which needed teacher´s help to finish the solution.

3 groups could not continue after the using of information: 1, 4, 5, 7, 8, 9, 12 and 14 (they put the colours of the houses well; each of these groups has the green – white position as in solution 1). The next step could be put the Dane and tea (information 3) to the 2^{nd} house (blue). The teacher helped them with question: *Which drink could be in which house?* After it the students could continue with the other given information.

One group had problem with the correct appointing of the colours. Teacher help with the colours of the houses:

Teacher: *What colour of house is exactly given?*

Student 1: *Blue, the second house because the Norwegian lives in the first house and lives next to the blue house.*

Teacher: *What we know about the other colours?*

Student 2: *The green house is next to the white house; on the left* (he put the green and white to position as in solution 1). *The Englishman lives in the red house.*

Teacher: *Where could we put the green and the white roofs?*

Student 2: *Here* (he´s pointing to the 3^{rd}, 4^{th} and 5^{th} houses).

Student 1: *The owner of the green house drinks coffee. They must be the 4^{th} and the 5^{th} houses.*

(The student put the green and the white roofs, and the red in the 1^{st} house and the yellow in the 3^{rd} house).

Teacher: *Why is the 1^{st} house the red and the 3^{rd} house the yellow? Could it be the other way around?*

Student 1: *No, it could not.*

(after 40 seconds)

Student 2: *Yes, it must be the other way around because the Norwegian lives in the 1^{st} and Englishman lives in the red house.*

2. Other 18 groups put to houses only 100% sure attributes and nationalities. The pairs following from the other information they placed next to the row of houses (Figure 7).

Seven from these groups got confused themselves by information 10: *The Blend smoker has a neighbour who keeps cats.* and 15: *The Blend smoker has a neighbour who drinks water.* (Figure 8) Three groups needed over 40 minutes only because this mistake.

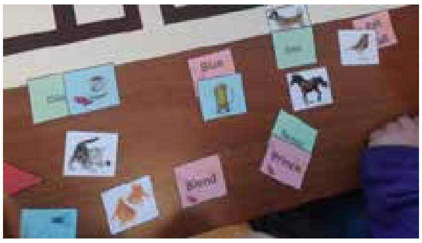

Fig. 7 The matching of the attributes

Fig. 8 The wrong matching of the attributes "water – Blend – cat" from information 10 and 15 (cat in the 3rd house)

Students did more mistakes during the solution:

1. They didn´t read the wording of the puzzle carefully as we understood from dialog:

Student 1: "*How do we know that Sweden has a dog?*"

Student 2: (point at information 2) *"It´s written here."*

Other group: the students put the Dunhill to blue house (we don´t understand why because information 7: *The owner of the yellow house smokes Dunhill*.)

2. When they line the colours up correctly, they tried to put other pairs of attributes and they exchange the colours – they ruined the correct colours.

3. They did wrong assumptions – milk in the 3rd house means that there must be the cat, water in the 1st house, there must be the fish.

We can say that students were interested in the solution of the puzzle. In the classroom could be heart students´ dialogs about the left side (which view – front or back) and other matching. They wrote us their opinions to this puzzle-based learning and it was positive. They will use puzzles in their future practise (this one too, but only with the best pupils as preparation for math competitions, in both, students and us opinion this puzzle is suitable for pupils of lower secondary education). We spoke about the educational aspect of the puzzle – there were cigarettes´ brands and alcoholic drinks – it must be change for non-alcoholic drinks and instead the cigarette brands we can use brands of cars.

When we told students that in Einstein´s opinion only 2% of population could solve this puzzle, they were happy for a moment. After that we told them that it is calculated only with people who use only mental solving – without drawing and paper. Their reaction was: *Without the paper houses and cars it will be impossible to solve*. We showed them some other methods for solving – different types of tables. They said that with these manipulative aids was the solution clear, funny and mainly unusual.

This experiment was very useful for us too. When we will do this puzzle with the next year students, we will change the wording of it. Some colours, nationalities, cigarette brands, drinks, and pets we can substitute for other ones. These do not change the logic of the puzzle.

3 Some zebra puzzles for different educational levels

After the solution of Neighbours puzzle we showed students some other zebra puzzles, which can be used at different school levels.

Puzzle 1[10]

There are three dogs: Alik, Rex and Max. We have three doghouses: green, blue and brown. There are three dog bowls: white, yellow and red.

a) Alik sleeps in blue house. Rex has a yellow bowl. Max sleeps in brown house and has a red bowl. Where does Rex sleep? Which bowl belongs to Alik?
b) Alik has a red bowl a does not sleep in brown house. Rex has a yellow bowl. Who has the white bowl, sleeps in blue house. Who sleeps where and which bowl is whose?
c) Max has neither white nor red bowl and sleeps in blue house. Alik does not sleep in green house and does not have a red bowl. Where sleeps Rex and which bowl is his?

Puzzle 2[11] (This puzzle was assigned as a task in mathematical competition Mathematical Kangaroo, in category Klokánek – pupils of 4th and 5th grade of primary school in Czech republic)

The street on the picture is called Coloured. You can find there a blue, red, yellow, pink and green house. The houses are numbered from 1 to 5. We know that:

· blue and yellow house are numbered with even numbers;
· red house is next only to the blue house;
· blue house is between green house and red house.

Which colour has the house numbered 3?

After these puzzles we can continue in the development of logical thinking with children for example with activity called *Smart heads*. Pavlovičová (2012) showed this activity as a tool for propaedeutic of propositional logic using instruction as: “*Choose head with cap which has a form of triangle and is dotted*.” “*Choose a head with cap which has a form of triangle or is not dotted*.”

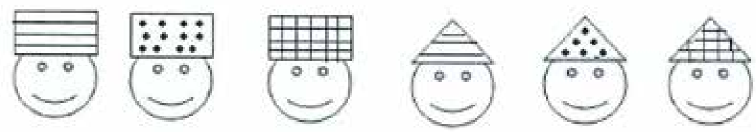

Fig. 9 The given heads with caps in activity Smart heads (Pavlovičová, 2012)

[10] KASLOVÁ, M.: Předmatematické činnosti v předškolním vzdělávání. Praha: Raabe, 2010. ISBN 978-80-86307-96-1
[11] http://www.glouny.cz/klokan/

Puzzle 3[12]

7 foreign delegates with their wives come together on a reception. They are sitting at a long table; their wives are sitting opposite to them. Each married couple eat and drink something else.

1. On the left side of the table sits the technologist.
2. The physicist sits between Mr Muller and Mr Fischer on the right side.
3. The husband of Mrs Helena is chemist.
4. Mrs Edita eats roast beef and drinks beer.
5. The physicist´s wife eats carp.
6. The husband of woman in yellow dress is a layer and sits on the other side of the table.
7. Mr Collins has on the right chemist, who is a passionate motorist.
8. The mathematician with his wife Eva sits in the middle.
9. Mr Blum, who plays tennis, sits next to technologist.
10. Mrs Dora has yellow dress and eats duck.
11. Mrs Joan sits next to Mrs Eva, who has a violet dress.
12. Mr Schwarz is husband of woman in blue dress; his neighbour eats goose.
13. The architect eats ham and drinks plum brandy. His neighbour Smith eats beefsteak and drinks cognac.
14. Mrs Marta sits next to Mrs Edita and is wearing orange dress and drink vodka.
15. The husband of the woman in grey dress likes climbing.
16. The wife of the doctor Mary drinks white wine; her husband likes horse race.
17. Mr White, who plays regularly golf, drinks red wine.
18. The husband of the woman in red dress loves rugby.

Who eats trout?
Whose hobby is fishing?
Who is the husband of the woman in red dress?

4 Conclusion

We believe that puzzle-based learning will take a place in the educational process in Slovak schools. We have to start to prepare university students - future teachers for this work because they do not use tasks in education which they could not solve. This part of mathematics – recreational mathematics - offers different problems and puzzles in which we can encourage students to find and learn some problem-solving methods and strategies. We can appeal them to discuss about solutions so we develop their competence of argumentation.

An additional utility of recreational mathematics is that it provides us a way to communicate mathematical ideas to the broader public. Many newspapers and professional magazines run regular mathematical puzzles – Martin Gardner´s columns were a major factor in the popularity of Scientific American and probably inspired more students to study mathematics than any other influence.

[12] PĚNČÍK, J. – PĚNČÍKOVÁ, J. Lámejte si hlavu. Praha: Prometheus, 1995. ISBN 80-7196-011-X

References

PRŮCHA, J. Moderní pedagogika (3. přepracované a aktualizované vydání). Praha: Portál, 2005. ISBN 80-7367-047-X

BOGGAN, M. et al. Using manipulatives to teach elementary mathematics. In: Journal of Instructional Pedagogies. Vol. 3, 2010. 8 p. ISSN 1941-3394 Cited 30. 5. 23013. Available at http://www.aabri.com/manuscripts/10451.pdf

BRUCKMAN, A. Can educational be fun? Paper presented at the Game Developer's Conference '99, San Jose, 1999. Cited on 30. 5. 2013. Available at http://www.cc.gatech.edu/ ~asb/papers/bruckman-gdc99.pdf

GAHÉR, F. Logika pre každého. Bratislava: IRIS, 2003. ISBN 80-89018-54-8

MICHALEWICZ Z. - MICHALEWICZ, M. Puzzle-based learning: An introduction to critical thinking, mathematics, and problem solving. Melbourne, 2008. ISBN 9781876462635

HANKIN, R. K. S. Recreational mathematics with R: introducing the magic package. 2013. Cited on 30. 5. 2013. Available at http://cran.r-project.org/web/packages/magic/vignettes /magicpaper.pdf

CHYTRÝ, V. Development of logical thinking using mathematical Games. Presentation of the article at XX. Czech – Polish –Slovak Mathematics School. Ústí nad Labem: UJEP, 2013.

KASLOVÁ, M.: Předmatematické činnosti v předškolním vzdělávání. Praha: Raabe, 2010. ISBN 978-80-86307-96-1

KULKARNI, D. Enjoying Math: Learning Problem Solving with KenKen® Puzzles. California: Pleasanton, 2012. ISBN 9778-0615702438

Math Kangaroo 2004 assignment (task 18 in category Klokánek). Cited on 30. 5. 2013. Available at http://www.glouny.cz/klokan/

MICHALEWICZ, Z. – FALKNER, N. – SOORIAMURTHI, R. Puzzle-Based Learning: an Introduction to Critical Thinking and Problem Solving. In: Decision Line, 2011.

MICHALEWICZ, Z. – FALKNER, N. – SOORIAMURTHI, R. Puzzle-Based Learning for Engineering and Computer Science. In: Computer, 2010. ISSN 0018-9162

PAVLOVIČOVÁ, G. et al.: Experimentujeme v elementárnej matematike. Nitra: UKF, 2012. ISBN 978-80-558-0126-1

PĚNČÍK, J. – PĚNČÍKOVÁ, J. Lámejte si hlavu. Praha: Prometheus, 1995. ISBN 80-7196-011-X

SINGMASTER, D. The Unreasonable Utility of Recreational Mathematics. 1993. Cited on 30. 5. 2013. Available at http://www.eldar.org/~problemi/singmast/ecmutil.html

TOMASI, Carlo. The Zebra Puzzle. 2006. Cited 30. 5. 2013. Available at http://www.cs.duke.edu/courses/spring06/cps102/notes/zebra.pdf

ŠVECOVÁ, V. – RUMANOVÁ, L. Supporting mathematical creativity in realistic surroundings. In: Generalization in mathematics at all educational levels. Rzeszow: Wydawnictwo Uniwersytetu Rzeszowskiego, 2012. P. 248-256 ISBN 978-83-7338-780-5

VIDERMANOVÁ, K. – KLEPANCOVÁ, M. Students´ investigation of Platonic solids – they are really only five. In: Mathematica IV. Ružomberok: KU v Ružomberku, 2012. P. 191-198 ISBN 978-80-8084-954-2

CONTENT AND LANGUAGE INTEGRATED LEARNING IN MATHEMATICS EDUCATION

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Abstract:
Content and language integrated learning (CLIL) in Czech schools is relatively new. In these days there are some projects aimed at CLIL implementation into Czech schools as well as at prospective teacher training in Faculties of Education for this teaching method.
In this article partial results of a project which is focused on findings about the use of CLIL in Czech schools, especially lower secondary schools, are presented. Given the still increasing trend of interdisciplinary relations and integration of subjects, the use of CLIL is an important element of the educational process. CLIL offers an efficient way of motivating pupils and great potential for their possible study abroad.
In the article training of prospective mathematics teachers in the Department of Mathematics, Faculty of Education, Palacký University in Olomouc, to use CLIL is presented.
Key words:CLIL, math, education
MESC: D30, D40

1 Introduction

Education is a major part of our lives. Without primary education, we would be unable to fully socialize. One of the contemporary trends is to migrate abroad and search job opportunities there. This means that language skills are required not only from the younger generations but from all citizens in general. That is why there are wider options of studying languages at elementary and secondary schools nowadays. It is no exception to study two world languages as compulsory subjects plus be able to learn more languages via optional seminars. The objective of this system is to prepare the present-day youth for being able to meet the requirements of the contemporary trend of the labour market.

While languages and other subjects are traditionally taught separately, the CLIL method enables the interconnection of specific subjects and foreign languages. Only rarely do teachers discuss subject-specific topics during language lessons; language courses are generally focused on traditional conversation topics and possibly the countries where the given language is spoken. Students thus lack not only subject-specific vocabulary but also an ability to discuss such technical topics. Therefore, the CLIL method is an important issue at present. It improves the ability of students to talk about conventional topics discussed during language lessons and also more specific topics belonging in non-language subjects.

2 Method CLIL

The abbreviation CLIL stands for "Content and Language Integrated Learning". Basically, it means an integration of content and language learning. Although this method is no novelty - the concept of CLIL was first officially used by David Marsh at a university in Finland in 1994 - it is relatively new in the Czech Republic. The official document which served to apply the CLIL method in the Czech environment was the "National Programme of Teaching Foreign Languages in the Czech Republic for the Period Between 2005 and 2008". It was prepared as a reaction to the Action Plan for the period between 2004 and 2006 issued by the European Commission. In 2009, the Ministry of Education, Youth and Sports issued a document entitled "Content and Language Integrated Learning in the Czech Republic". Since then, several projects have been ongoing in the Czech Republic, aiming at the introduction of CLIL into Czech schools and training teachers to use the method properly. Some of the projects are conducted at the Department of Mathematics of the Faculty of Education of Palacký University in Olomouc as part of the training of future teachers.

3 CLIL – advantages and disadvantages

Like every method, CLIL has its advantages and disadvantages. One of the major disadvantages of the CLIL method lies in its demands. Not only is it more time-consuming as regards the necessary preparation for classes, but it is also more demanding in terms of teachers' language skills. The preparation of materials is also difficult; however, their availability is increasing via various projects, portals and trainings. It is also necessary to choose activities that are sufficiently self-explanatory and reasonably difficult. Another disadvantage lies in the risk of demotivation. Students may be demotivated especially in two cases. If a student is experiencing difficulties in a subject taught in their mother tongue, it is highly unlikely that the student will be successful in a foreign language. The second reason may be low self-confidence as regards language skills. Both the above cases may be solved via suitable positive motivation.

If teachers manage to overcome these obstacles, the CLIL method may bring many advantages. One of the major advantages is the improvement of students' communication skills in a foreign language. The method also helps students to expand their subject-specific vocabulary for which there is generally not enough time in conventional language courses. Students are thus prepared for their future studies abroad, since they are more experienced in subject-specific communication in a foreign language and need not fear any communication blocks in a foreign country. The absence of such fears is very useful as regards studying and also possible job seeking abroad. Students are more likely to find jobs in the future, not only in the Czech labour market but also abroad. The CLIL method does not necessarily have to be used for the entire duration of each class; it may be applied only to certain parts of the lesson, for example "mathematical warm-ups" or revisions at the end of the lesson. This brings an effective innovation and improves the attitude of students towards the given subject. Students may also be more motivated in terms of acquiring new knowledge (Hofmannová, Novotná 2003).

There are several prerequisites for a successful application of the CLIL method. One of the most important prerequisites for teachers who want to use the CLIL method is not only a sufficient subject-specific knowledge but also sufficient language skills. At the same time, it is necessary to consider the topics to be taught in the foreign language. Such topics should be self-explanatory - if there is the possibility of graphic representation, it is more likely that students will understand the lesson. It is important to expand students' vocabulary for example via vocabulary lists handed out to students. Of course, students need to be positively motivated. They should not feel that any task is beyond their skills, nor should they fear failure. On the other hand, teachers should motivate them for example through a positive feedback.

4 CLIL at schools

As part of the initial stage of the project entitled Student Grant Competition of Palacký University, I wanted to learn the current status of the CLIL method at elementary schools and grammar schools in the Olomouc and South Moravian Regions. I used questionnaires sent via e-mail to the managements of the above schools. The objective was to find out whether the schools use the CLIL method, and if they do, in which subjects. The rate of return was between 16% and 18% in both regions. The following information was found. Out of the total number of replies regarding the use of the CLIL method in the Olomouc Region, 10% were positive. These replies included the current use of the method as well as the use in the previous years or the preparation of foreign language integration for the next term. In most cases, the foreign language intended for integration into other subjects was English, only in one case it was French. Almost all subjects were represented that were taught or to be taught in the foreign language and no subject significantly prevailed over others. In the South Moravian Region, the situation was different. 13% of replies were positive. Similarly, the English language was dominant and a wide scope of subjects was represented. A major difference consisted in the dominance of some subjects. The most frequent were Maths, Arts, Music, Civics and Physical Education. Among other frequent subjects were Informatics, Social Sciences, Natural Science and History. These findings indicate that the CLIL method is not that widespread yet - there is a frequent fear that such lessons are too difficult and the preparation for them too demanding. What is encouraging, however, is the fact that this modern method is gradually being introduced into our schools, encompassing a wide scope of subjects.

5 CLIL at Department of Mathematics

As has already been mentioned, the use of the CLIL method is also taught to future teachers. One of the examples is the Department of Mathematics of the Faculty of Education of Palacký University in Olomouc, specifically the compulsorily optional course of Mathematical Terminology in English. This course is taught using the CLIL method and is focused on three main blocks - algebra, mathematical analysis and geometry. Each of the blocks uses study support in the form of a workbook and the above mentioned vocabulary list of the most important words and phrases. Future teachers thus have the opportunity to experience lessons in a foreign language and find out the advantages and disadvantages of this method. And of course, they can use their personal experience in their future teaching practice.

References

Content and Language Integrated Learning v ČR. [online]. [cit. 2013-03-01]. Dostupné z: http://www.msmt.cz/vzdelavani/content-and-language-integrated-learning-v-cr

HANUŠOVÁ, VOJTKOVÁ. CLIL v české praxi. In Odborná regionální konference CLIL v ČR a zahraničí. 2012.

HOFMANNOVÁ, M. and NOVOTNÁ, J.. CLIL – Nový směr ve výuce. Cizí jazyky, roč. 46, 2002/2003, číslo 1, p. 5-6. ISSN 1210-0811.

Integrovaná výuka cizího jazyka a odborného předmětu – CLIL: sborník z konference. Praha: Výzkumný ústav pedagogický v Praze, 2011. ISBN 978-80-87000-85-4.

Národní plán výuky cizích jazyků. [online]. [cit. 2012-10-06]. Dostupné z: http://aplikace.msmt.cz/PDF/JT010NPvyukyCJnaNet.pdf

NOCAR, D. Inovovaný koncept matematické složky profesní přípravy učitelů primární školy na Pedagogické fakultě Univerzity Palackého v Olomouci. In Acta Universitatis Palackianae Olomucensis, Facultas Paedagogica, Mathematica VII. Olomouc: Univerzita Palackého, 2010. ISSN 0862-9765.

POKRIVČÁKOVÁ, S.. Modernization of teaching foreign languages: CLIL, inclusive and intercultural education. 1. vyd. Brno: Masarykova univerzita, 2010, 219 s. ISBN 978-802-1052-949.

Seznamte se s CLILem. Praha: Národní ústav pro vzdělávání, školské poradenské zařízení a zařízení pro další vzdělávání pedagogických pracovníků (NÚV), divize VÚP, 2011. ISBN 978-80-87063-52-1.